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**PSEUDO-COMPLEX GENERAL RELATIVITY:
SCHWARZSCHILD, REISSNER-NORDSTRÖM AND KERR
SOLUTIONS**

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The pseudo-complex General Relativity (pc-GR) is further considered. A new projection method is proposed. It is shown, that the pc-GR introduces automatically terms into the system which can be interpreted as dark energy. The modified pseudo-complex Schwarzschild solution is investigated. The dark energy part is treated as a liquid and possible solutions are discussed. As a consequence, the collapse of a large stellar mass into a singularity at $r = 0$ is avoided and no event-horizon is formed. Thus, black holes don't exist. The resulting object can be viewed as a gray star. It contains no singularity which emphasizes, again, that it is not a black hole. The corrections implied by a charged

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large mass object (Reissner- Nordström) and a rotating gray star (Kerr) are presented. For the latter, a special solution is presented. Finally, we will consider the orbital speed of a mass in a circular orbit and suggest a possible experimental verification.

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1. Introduction

General Relativity (GR) is an extremely successful theory, which up to now is confirmed by a great number of experiments. Nevertheless, it contains a singularity for highly dense small objects (Schwarzschild singularity) and the appearance of singularities in any field theory shows, that its correct form has not yet been found. Einstein's GR predicts the existence of *black holes*, which imply singularities at the center of great masses and the appearance of an event horizon at the Schwarzschild radius, r_s . Due to this reason (and some others which we will mention later on) several groups tried to extend GR such that the new theory does not contain the mentioned undesired property.

In an attempt to extend GR, for example, Einstein in ^{1,2} substituted the real space-time variables by complex ones, an idea which was continued in ^{3,4} and called it *complex GR*. Others took the idea of M. Born ^{5,6}, which treats the space-time coordinates and its conjugate momenta on equal footing. Indeed in Quantum Mechanics coordinates and momenta can be interchanged through a canonical transformation. Thus, there might be hope to connect GR to Quantum Mechanics. Our proposal consists in the redefinition of the length square element, which approaches the one of standard GR for large masses and distances, while for small distances the coordinates and momenta appear symmetrically in the length square element. This extension introduces a maximal acceleration, as was shown in ⁷. In ^{8,9,10,11,12,13,14,15,16,17} the consequences of this approach were investigated. In another attempt to extend GR, hyberbolic (another name for pseudo-complex) coordinates were introduced ^{18,19,20,21}. In ²² we extended GR to pseudo-complex coordinates. The new pseudo-complex General Relativity (pc-GR) contains the former mentioned theories. The pseudo-imaginary component is associated with the components of the tangent vector (four momentum) at a given space-time point. Due to dimensional reasons, a minimal length scale (l) is introduced into the theory. It corresponds to a maximal acceleration. The pc-Schwarzschild solution was obtained, neglecting all corrections due to the minimal length scale l . Due to the pc-description an additional contribution appears in the energy-momentum tensor. One can say, that the modified field equations can be interpreted (within the old Einsteinian GR) as dark energy: The additional terms play the role of a repulsive dark energy. The energy accumulates around a central mass and halts the collapse of a big star. No singularity and no event horizon forms. Thus, no black hole exists in this theory. This is an important achievement! The new highly dense objects

may be called *gray stars*. In ²³ the pc-Robertson-Walker model of the universe was proposed. Also there, terms, which can be interpreted as “dark energy” appear automatically, and new solutions are obtained, like a universe which at large time tends to a constant acceleration or diminishes its acceleration, approaching zero.

In this paper we refer for an introduction to pseudo-complex variables to ^{22,23} and ²⁴. An extensive mathematical treatment of pseudo-complex coordinates can be found in ^{25,26}.

We revisit the initially proposed pseudo-complex extension of GR in ^{22,23}, where the projection method suffered from conceptual difficulties: for example the projected metric did not represent a tensor. In section 2 we will investigate an alternative projection method which will result in a real metric with tensorial properties. In section 3 we will apply it to the pc-Schwarzschild solution. As a new ingredient, the dark energy is treated as a fluid and the differential equations are derived in sub-section 3.3. Both the pressure and density have to obey these equations. Some approximate solutions are discussed. In section 4 a charged gray star, i.e. the pc-Reissner-Nordström solution, is investigated. Section 5 deals with the pc-Kerr problem. The equations of motion will be formulated. A specific solution will also be discussed, which shows similarities to the standard solution of Einsteins GR, with the essential difference that, again, black holes do not exists. The results can be used for further studies to find possible analytic solutions for a rotating gray star.

We also discuss in section 6 possible experimental verifications for further testing our theory. In particular the orbital speed of a mass in a circular orbit around a gray star is considered.

Section 7 contains the conclusions.

2. Reformulation of the pseudo-complex General Relativity

Let us first briefly remind on some key properties of pc-variables: A pc coordinate X is given by $X = X_R + IX_I$, where X_R is the pseudo-real and X_I its pseudo-imaginary component. Contrary to complex variables, the $I^2 = 1$. This is, at first, astonishing. But remember that e.g. for the Pauli-matrices $\hat{\sigma}_i$ one has $\hat{\sigma}_i^2 = 1$. Thus one can think of pseudo-complex coordinates as matrix-coordinates. Have in mind a similar situation, when Dirac introduced instead of $E = \pm\sqrt{(pc)^2 + (m_0c^2)^2}$ the matrix $E = \sum_i \gamma^i \hat{p}_i$. This lead from the Klein-Gordon equation to the Dirac equation, which contained spin and was the basis for a model of the vacuum and for creation of particle-antiparticle pairs.

A pseudo-complex conjugate variable is defined as $X^* = X_R - IX_I$. Instead of this representation one can express the variables as a linear combination of $\sigma_{\pm} = \frac{1}{2}(1 \pm I)$, i.e., $X = X_+\sigma_+ + X_-\sigma_-$, with $X_{R/I} = \frac{1}{2}(X_+ \pm X_-)$ and $\sigma_{\pm}^2 = \sigma_{\pm}$, $\sigma_+\sigma_- = 0$. The last property implies that the pseudo-complex variables contain a zero divisor ^{25,26}, which produces very important changes in the structure of GR. Variables, which are within the zero divisor, satisfy the property $|X|^2 = X^*X =$

0. These variables can be expressed either as $X = X_+\sigma_+$ or $X = X_-\sigma_-$ and they constitute the *zero divisor* denoted by \mathcal{P}^0 . One important property of the use of pseudo-complex variables is that *any mathematical application can be done independently within the σ_+ and the σ_- component*. For example, the multiplication of two functions $F(X) = F_+(X_+)\sigma_+ + F_-(X_-)\sigma_-$ and $G(X) = G_+(X_+)\sigma_+ + G_-(X_-)\sigma_-$ is just $F(X)G(X) = F_+(X_+)G_+(X_+)\sigma_+ + F_-(X_-)G_-(X_-)\sigma_-$. This property will enable us later to define two different metric tensors, one in the σ_+ and the other one in the σ_- sector and that in each sector we can define a standard GR. The X_\pm will be denoted as the *zero-divisor components* of X .

In a pc-theory, the variational principle has to be modified, such that

$$\delta S \in \mathcal{P}^0 \quad , \quad (1)$$

where \mathcal{P}^0 represents the zero divisor. One can show^{22,24} that this change in the variational principle is necessary because otherwise one would generate two independent theories, one in the σ_+ and the other one in the σ_- sector of the action. In a pc-theory one has then to worry on how to connect both components, when real results are projected out.

For convenience we choose the variation of the action proportional to σ_- and note that using σ_+ results in the same description: only the σ_+ with the σ_- component of the theory are interchanged.

Extending a theory to a pseudo-complex version will, in general, introduce important simplifications. For example, as shown in^{24,27,28} the pseudo-complex extension of a *linear* Field Theory leads to a regularized theory á la Pauli-Villars, *maintaining the linear description* of the field theory, while not applying the pc-extension the Lagrange density is highly non-linear.

In a pc-field theory, the projection to the real part is obtained by noting that the pseudo-complex extension of the Lorentz group (similar for the Poincaré group) is given by^{29,30}

$$SO_+(3,1) \otimes SO_-(3,1) \supset SO(3,1) \quad , \quad (2)$$

where finite transformations in the $SO_+(3,1) \otimes SO_-(3,1)$ groups and the projection to $SO(3,1)$ are given by

$$e^{\omega^{\mu\nu} L_{\mu\nu}} \rightarrow e^{\omega_R^{\mu\nu} L_{\mu\nu}^R} \quad . \quad (3)$$

Here, $\omega^{\mu\nu} = \omega_R^{\mu\nu} + I\omega_I^{\mu\nu}$ are the pseudo-complex transformation parameters and $L_{\mu\nu} = X_\mu P_\nu - X_\nu P_\mu$ are the pseudo-complex generators. In the projection of $L_{\mu\nu}$ to $L_{\mu\nu}^R$, the coordinates X_μ are substituted by x_μ and the momenta P_ν by p_ν . The finite transformation is a function $f(\omega_{\mu\nu}, X_\mu, P_\mu)$ in the pseudo-complex coordinates (X_μ) , momenta (P_μ) and transformation parameters $(\omega_{\mu\nu})$. As can be seen, the function f is not mapped to $\frac{1}{2}(f + f^*) = \frac{1}{2}(f_+ + f_-)$, which is also real but not

necessary the same function anymore, but rather the *arguments* of the function are mapped to their real values, yielding *the same function* but with real arguments. To illustrate that, we consider the example of a function $f = \frac{1}{R+A}$, where A has no σ_+ -part, e.g. $A = A_- \sigma_-$. $\frac{1}{2}(f_+ + f_-)$ is then given by

$$\frac{1}{2}(f_+ + f_-) = \frac{1}{2} \left(\frac{1}{R_+} + \frac{1}{R_- + A_-} \right) = \frac{1}{2} \left(\frac{R_- + A_- + R_+}{R_+(R_- + A_-)} \right) = \frac{r + A_{\text{Re}}}{r^2 + 2rA_{\text{Re}}} \quad (4)$$

On the other hand, if we map the arguments of f we receive

$$f = \frac{1}{R+A} \rightarrow \frac{1}{r + A_{\text{Re}}} \quad (5)$$

This example has to be kept in mind, when we define the projection to the real space in the pc-extension of GR, i.e., the proposed projection will substitute in any function of the pseudo-complex coordinates, momenta and parameter values to *the same function* but now in the real coordinates, momenta and parameter values.

There is another simplification provided by the pc-description: As shown in ²⁴, the field equation for a scalar boson field is obtained from the Lagrangian density $\frac{1}{2}(D_\mu \Phi D^\mu \Phi - M^2 \Phi^2)$, where Φ is the pc-boson field, $M = M_+ \sigma_+ + M_- \sigma_-$ is a pc-mass and D_μ a pc-derivative. The propagators of this theory are the ones of Pauli-Villars, which already are regularized. One obtains the same propagator in the standard theory, with non-pc scalar field, using the Lagrange density $-\frac{1}{(M_+^2 - M_-^2)} \phi (\partial_\mu \partial^\mu + M_+^2) (\partial_\mu \partial^\mu + M_-^2) \phi$, where ϕ is now a real valued function, M_+ is identified with the physical mass m and $M_- \gg M_+$ with the regularizing mass. Note, that this theory is highly non-linear while the pc-description is linear. This indicates that a pc-description can substantially simplify the structure of the theory.

In ^{15,16} Moffat et al. proposed a possible extension of GR by introducing a non-symmetric metric. In ³¹ the symmetric part of the metric was associated to the pseudo-real component of a pseudo-complex metric (called in ³¹ hyper-complex) and the antisymmetric metric was associated to the pseudo-imaginary component. Investigating the linearized limit for small fields, it was shown that the Lagrangian separates into two parts, one depending only on the symmetric and the other one on the anti-symmetric components. It was proven that due to $I^2 = 1$ only standard GR and the pseudo-complex extension of it do not contain ghost solutions, thus only these theories are physical. If $g_{\mu\nu}^{\text{M}}$ denotes the metric of Moffat, the zero divisor components are given by $g_{\mu\nu}^{\text{M}}$ for the σ_+ component and $(g_{\mu\nu}^{\text{M}})^T$ (T stands for *transposed*) for the σ_- component. These components are different because Moffat assumes a non-symmetric metric. If the metric would be symmetric, then both components would be equal. In contrast to this, we will consider for the two zero-divisor components of the metric *two different but symmetric metrics*. This will not change the main conclusions given in ³¹: In each zero-divisor component we have a standard GR with a symmetric metric and according to ³¹ both contain only

physical solutions in the weak field limit. Afterwards we have to project to real solutions.

In pc-GR ²² the space-time coordinates X^μ ($\mu = 0,1,2,3$) have pseudo-real components, which are identified with the real space-time components x^μ , while the pseudo-imaginary components are given by $(l/c)u^\mu$, proportional to the tangent vector at a given space-time point, which can be identified in case of a world line of a particle as its four-velocity at a given point of the world line. The factor l has the units of a length and is introduced for dimensional reasons. This automatically introduces a minimal length scale (l) into the theory which implies a maximal acceleration ⁷. The consequences of that have not been investigated yet in a pc-GR, but we suspect that it will be along the lines as discussed in ³². The projection, as given in (3), then is equivalent to setting $l = 0$, or accelerated systems are excluded.

Note, that in most publications units are used where $c = 1$ (as done by us also in ²²). Now, we opted to use c with its MKS units for practical reasons. In the MKS units, the l/c has the unit of time.

In pc-GR the metric has a pseudo-real and a pseudo-imaginary component, or the zero-divisor components,

$$\begin{aligned} g_{\mu\nu} &= g_{\mu\nu}^R + I g_{\mu\nu}^I \\ &= g_{\mu\nu}^+ \sigma_+ + g_{\mu\nu}^- \sigma_- \quad , \end{aligned} \quad (6)$$

are both assumed to be symmetric, i.e. we do not follow Moffat's proposal of a non-symmetric metric, though our theory can be extended to it. Furthermore,

$$g_{\mu\lambda} g^{\lambda\nu} = \delta_\mu^\nu \quad . \quad (7)$$

The σ_\pm components of the metric are related to their pseudo-real and pseudo-imaginary components via

$$\begin{aligned} g_{\mu\nu}^R &= \frac{1}{2} (g_{\mu\nu}^+ + g_{\mu\nu}^-) \\ g_{\mu\nu}^I &= \frac{1}{2} (g_{\mu\nu}^+ - g_{\mu\nu}^-) \quad . \end{aligned} \quad (8)$$

Choosing $g_{\mu\nu}^+ = g_{\mu\nu}^M$ and $g_{\mu\nu}^- = (g_{\mu\nu}^M)^T$, leads to Moffat's proposal, i.e., the "-" component of the metric is the transposed of the "+" component of the metric. This gives a connection between both components, which determines the system. In this contribution, however, the metric is assumed to be symmetric in both components and a connection between them has to be defined yet.

Due to the division into pseudo-divisor components we can define a standard GR in both sectors, but with a different symmetric metric. Thus, parallel displacements, connections, four derivatives, etc., are defined in *exactly the same way* as in standard GR. Also, the four-derivative of the metric is zero, permitting the definition of a universal pseudo-complex metric (for details, see ²²).

Due to the new variational principle, the modified Einstein equation is given by

$$\mathcal{R}^{\mu\nu} - \frac{1}{2}g^{\mu\nu}\mathcal{R} = -\frac{8\pi\kappa}{c^2}T_-^{\mu\nu}\sigma_- \quad , \quad (9)$$

where we have chosen the right hand side to be proportional to σ_- . The notation on the right hand side is intentional, because the requirement that a function is within the zero-divisor introduces in the σ_- component a function, which can be associated to the energy of a yet to be determined field. As shown in ²² this term is responsible for the disappearance of the black hole (and the singularity), producing an anti-gravitational effect for high central mass density, which halts the collapse of the mass distribution.

In ²² a projection method was proposed, which we will briefly explain and we shall also outline the problems related to it. The proposed projection is performed, requiring that the pseudo-complex length element $d\omega^2$ is real, which leads to

$$\begin{aligned} d\omega^2 &= g_{\mu\nu}DX^\mu DX^\nu \\ &= g_{\mu\nu}^0 \left(dx^\mu dx^\nu + \left(\frac{l}{c}\right)^2 du^\mu du^\nu \right) + 2\left(\frac{l}{c}\right) h_{\mu\nu} dx^\mu du^\nu \quad , \end{aligned} \quad (10)$$

with

$$\begin{aligned} g_{\mu\nu}^0 &= \frac{1}{2}(g_{\mu\nu}^+ + g_{\mu\nu}^-) \\ h_{\mu\nu}^0 &= \frac{1}{2}(g_{\mu\nu}^+ - g_{\mu\nu}^-) \quad . \end{aligned} \quad (11)$$

Note that, the "metric" $g_{\mu\nu}^0$ is not a tensor and, therefore, can not lower the index of x^μ nor of u^μ (see Eq. (12) below). Second, using the orthogonality relation $g_{\mu\lambda}^\pm g_\pm^{\lambda\nu} = \delta_\mu^\nu$ we are led to

$$g_{\mu\lambda}^0 g_0^{\lambda\nu} + h_{\mu\lambda} h^{\lambda\nu} = \delta_\mu^\nu \quad . \quad (12)$$

This shows that the matrix $g_0^{\mu\nu}$ is not the inverse of $g_{\mu\nu}^0$.

Due to these properties, we propose a projection as the one applied in the pc-field theory, mentioned above. Once a pseudo-complex function in terms of the coordinates and momenta is given, its projection is obtained by substituting all pseudo-complex variables, momenta (velocities) and parameters of the theory by their pseudo-real parts. Also possibly appearing parameters in these functions, like the transformation parameters $\omega^{\mu\nu}$ in a finite Lorentz transformation, have to be substituted by their pseudo-real part. This will be applied here too and examples will be given, like the pc-Schwarzschild, the Reissner-Nordström and the Kerr solution.

In general, let us denote by G_\pm the geometrical space of the σ_\pm component. The zero-divisor metric components are given by $g_{\mu\nu}^\pm$. The two geometrical spaces

commute due to the property of the σ_{\pm} operators. This we denote via $G_+ \otimes G_-$ which is finally reduced by the projection to the real geometrical space G , i.e.

$$G_+ \otimes G_- \supset G \quad . \quad (13)$$

Again, the projection corresponds to set within the metric all contributions, which are proportional to powers in l , to zero. This leads to a metric, which does not depend on the acceleration. The case where the contributions of higher powers in l are important to the structure of the metric in certain areas of the space-time, i.e., the case where the metric depends on the acceleration of a system, has still to be investigated.

For the metric, the final projection is achieved by

$$g_{\mu\nu}(X, \mathcal{A}) \rightarrow g_{\mu\nu}(x, \mathcal{A}^R) \quad , \quad (14)$$

where X, x is a shorthand notation for $(X^\lambda), (x^\lambda)$ and $\mathcal{A}, \mathcal{A}^R$ for a set of pseudo-complex (\mathcal{A}_α) and pseudo-real parameters (\mathcal{A}_α^R) respectively.

It is interesting to know the first important contributions of l to the length element. In fact the first object is the metric itself. It can be expanded in powers of $(l/c)u^\mu$. However, the norm of u^μ is always smaller or equal to the speed of light. Assuming that the minimal length scale is of the order of the Planck length, implies that these contributions can be safely neglected and the metric can still be approximated by (14). However, this is not the case for $DX^\mu DX^\nu$ in the length element. There a term $(l/c)^2 du^\mu du^\nu$ appears, i.e., it depends on the *change* of the four velocity, i.e. on the acceleration. If one approaches situations with the maximal acceleration c^2/l , then the term $(l/c)^2 du^\mu du^\nu$ is of the order of $l^0 = 1$ and can not be neglected. In normal situations, where the acceleration is small, the case considered here, then also this term can be neglected and the length element is of the same form as in standard GR. For large accelerations, the length element approaches the form as used by other theories, like the one proposed by Born^{5,6} and Caianiello⁷. This is the reason why we mention it here, because, obviously these theories are a special limit of ours. We will not further elaborate on this but refer to first attempts in this direction³².

With this motivation, for the length element squared we have

$$\begin{aligned}
d\omega^2 &= g_{\mu\nu}(X, P)DX^\mu DX^\nu \\
&= g_{\mu\nu}(X, P) \left(dx^\mu dx^\nu + \left(\frac{l}{c}\right)^2 du^\mu du^\nu \right) + g_{\mu\nu}(X, P) 2 \left(\frac{l}{c}\right) Idx^\mu du^\nu \\
&\rightarrow g_{\mu\nu}(x, p) \left(dx^\mu dx^\nu + \left(\frac{l}{c}\right)^2 du^\mu du^\nu \right) + g_{\mu\nu}(x, p) 2 \left(\frac{l}{c}\right) Idx^\mu du^\nu \\
&\rightarrow g_{\mu\nu}(x, p) \left(dx^\mu dx^\nu + \left(\frac{l}{c}\right)^2 du^\mu du^\nu \right) .
\end{aligned} \tag{15}$$

After mapping X and P to x and p (including the mapping of parameters) in the third line, in the last step in (15) the pseudo imaginary part of the length square element, appearing in the second line, has been set to zero, due to the condition that the length element has to be pseudo-real. It is interesting to note that the condition of disappearance of the pseudo-imaginary part in the length element (15) leads to

$$g_{\mu\nu}(x, p)dx^\mu du^\nu = 0 \quad , \tag{16}$$

which is nothing but the dispersion relation of a particle. Normally, this dispersion relation is set to zero by hand (see, for example, ³²). In our procedure it is a consequence of the condition that $d\omega^2$ is pseudo-real and it represents a subsidiary condition.

Note, that the resulting $d\omega^2$ in (15) is equal to the one used by M. Born and related theories (setting $c = 1$).

3. The pseudo-complex Schwarzschild solution and fluid description for the dark energy

In this section the pc-Schwarzschild solution will be revisited. Several steps are directly copied from ³³ but reformulated within the pseudo-complex language.

The pseudo-complex length element square for the Schwarzschild solution has the form

$$d\omega^2 = e^\nu (DX^0)^2 - e^\lambda (DR^0)^2 - R^2 [(D\theta)^2 + (\sin\theta)^2 (D\phi)^2] \quad , \tag{17}$$

The Einstein equation determines the explicit structure of ν and λ .

In ²² the additional, arbitrary constraint $\mathcal{R} = 0$ for the curvature scalar was imposed. This, however, led to a unnatural large contribution to the metric tensor, excluded by experiment. Therefore, this condition is skipped here.

The components $\mathcal{R}_{\mu\nu}$ of the Ricci tensor in the Einstein equation reduce to $\mathcal{R}_{\mu\nu}\epsilon\mathcal{P}^0$. $\mathcal{R}_{\mu\nu}$ is the pc-Ricci tensor. As defined in ²², we have

$$\begin{aligned}
\mathcal{R}_{00} &= -\frac{1}{2}e^{\nu-\lambda}\xi_0\sigma_- \\
\mathcal{R}_{11} &= \frac{1}{2}\xi_1\sigma_- \\
\mathcal{R}_{22} &= \xi_2\sigma_- \\
\mathcal{R}_{33} &= \xi_3\sigma_- = \xi_2\sin^2\theta\sigma_- \quad .
\end{aligned} \tag{18}$$

Denoting with a prime the derivation with respect to R , the former condition $(\nu' + \lambda') = 0$ of the standard GR changes now to

$$(\nu' + \lambda') = \frac{1}{2}R(\xi_0 - \xi_1)\sigma_- \quad . \tag{19}$$

In ²² it was assumed for simplicity that $\xi_0 = \xi_1$. This, however, led together with $\mathcal{R} = 0$ to corrections in the metric of order $\frac{1}{R^2}$, which are excluded by the experiment ³⁴. *This led us to reconsider the pc-Schwarzschild solution, skipping the above conditions.*

We rewrite the Einstein equation into

$$\mathcal{R}_\mu^\nu - \frac{1}{2}g_\mu^\nu\mathcal{R} = \Xi_\mu^\nu\sigma_- \quad . \tag{20}$$

where we used the abbreviation

$$\Xi_\mu^\nu = -\frac{8\pi\kappa}{c^2}T_\mu^\nu \quad , \tag{21}$$

with T_μ^ν representing the components of the energy-momentum tensor. We also rewrote the equation into upper and lower index notation.

Because the Ricci tensor is diagonal, the following definition is also used

$$\Xi_\mu^\mu = \Xi_\mu \tag{22}$$

and from that we obtain a relation between Ξ_μ and ξ_μ , i.e.,

$$\begin{aligned}
-\frac{1}{4}e^{-\lambda_-}\xi_0 + \frac{1}{4}e^{-\lambda_-}\xi_1 + \frac{\xi_2}{R_-^2} &= \Xi_0 \\
\frac{1}{4}e^{-\lambda_-}\xi_0 - \frac{1}{4}e^{-\lambda_-}\xi_1 + \frac{\xi_2}{R_-^2} &= \Xi_1 \\
\frac{1}{4}e^{-\lambda_-}(\xi_0 + \xi_1) &= \Xi_2 \\
\Xi_3 &= \Xi_2 \quad .
\end{aligned} \tag{23}$$

In principle, we can abolish the use of the ξ_μ functions and keep only the Ξ_μ . However, because we introduced the ξ_μ functions in ²² and want to compare with the results there, we will keep the ξ_μ functions and express Ξ_μ in terms of ξ_μ .

The equations in (23) are resolved for ξ_μ , leading us to

$$\begin{aligned}\frac{2\xi_2}{R_-^2} &= \Xi_0 + \Xi_1 \\ \frac{1}{2}e^{-\lambda_-}(\xi_0 - \xi_1) &= \Xi_1 - \Xi_0 \\ \frac{1}{4}e^{-\lambda_-}(\xi_0 + \xi_1) &= \Xi_2 \quad .\end{aligned}\tag{24}$$

Multiplying in a first step the last equation with 2 and adding to it the second equation and subtracting in a second step the second equation from the last one, we obtain

$$\begin{aligned}e^{-\lambda_-}\xi_0 &= 2\Xi_2 + \Xi_1 - \Xi_0 \\ e^{-\lambda_-}\xi_1 &= 2\Xi_2 - \Xi_1 + \Xi_0 \\ \frac{2\xi_2}{R_-^2} &= \Xi_0 + \Xi_1 \quad .\end{aligned}\tag{25}$$

Obviously the last equation in (25) is a repetition of the first equation in (24).

Now, we use for $T_\mu^\nu = \text{diag}(\rho, -\frac{p}{c^2}, -\frac{p}{c^2}, -\frac{p}{c^2})$ the expression for an ideal fluid/gas³³ together with (21), which gives

$$\begin{aligned}\Xi_0 &= -\frac{8\pi\kappa}{c^2}\rho \\ \Xi_k &= \frac{8\pi\kappa}{c^2}\frac{p}{c^2} \quad , \quad (k = 1, 2, 3) \quad .\end{aligned}\tag{26}$$

Substituting this into equation (25) gives

$$\begin{aligned}\frac{2\xi_2}{R_-^2} &= -\frac{8\pi\kappa}{c^2}\left(\rho - \frac{p}{c^2}\right) \\ e^{-\lambda_-}\xi_0 &= \frac{8\pi\kappa}{c^2}\left(3\frac{p}{c^2} + \rho\right) \\ e^{-\lambda_-}\xi_1 &= -\frac{8\pi\kappa}{c^2}\left(\rho - \frac{p}{c^2}\right) \quad ,\end{aligned}\tag{27}$$

which immediately tells us that

$$e^{-\lambda_-}\xi_1 = \frac{2\xi_2}{R_-^2} \quad .\tag{28}$$

This result is independent of the assumption that $\xi_0 = \xi_1$! (28) can also be obtained by noting that $\Xi_1 = \Xi_2$ (see (26)) and using (23).

3.1. Solving the Einstein equation

We will concentrate on the σ_- component, because the σ_+ component is the same as in ³³.

A repeated differentiation of (19) leads to

$$\nu''_- = -\lambda''_- + \frac{1}{2}(\xi_0 - \xi_1) + \frac{R_-}{2}(\xi'_0 - \xi'_1) \quad . \quad (29)$$

Copying the steps of ²², Eqs. (52)

$$\begin{aligned} \mathcal{R}_{00} &= -\frac{e^{\nu-\lambda}}{2} \left(\nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} + \frac{2\nu'}{R} \right) \\ \mathcal{R}_{11} &= \frac{1}{2} \left(\nu'' + \frac{\nu'^2}{2} - \frac{\lambda'\nu'}{2} - \frac{2\lambda'}{R} \right) \\ \mathcal{R}_{22} &= (e^{-\lambda}R_-)' - 1 \\ \mathcal{R}_{33} &= \sin^2 \theta \left[(e^{-\lambda}R_-)' - 1 \right] \quad . \end{aligned} \quad (30)$$

and (53)

$$\begin{aligned} \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' + \frac{2\nu'}{R_-} &= \xi_0\sigma_- \\ \nu'' + \frac{1}{2}\nu'^2 - \frac{1}{2}\lambda'\nu' - \frac{2\lambda'}{R_-} &= \xi_1\sigma_- \quad . \end{aligned} \quad (31)$$

and restricting to the σ_- component only, we obtain from the equation $\mathcal{R}_{11}^- = \frac{1}{2}\xi_1$ the following one

$$\nu''_- + \frac{1}{2}\nu'^2_- - \frac{1}{2}\lambda'_-\nu'_- - \frac{2\lambda'_-}{R_-} = \xi_1 \quad . \quad (32)$$

This is the same as in ²². Because in general $\xi_0 \neq \xi_1$, in the equation $R_{22}^- = \xi_2$ (see (18)) an additional term appears in R_{22}^- , namely $R_-e^{-\lambda_-} \left(\frac{\nu'_- + \lambda'_-}{2} \right)$, which is proportional to $(\lambda'_- + \nu'_-)$, not equal to zero. Its origin is shown in ³³, Eq. (6.44). Taking this term into account changes the equation $R_{22}^- = \xi_2$ to

$$[R_-e^{-\lambda_-}]' - 1 + R_-e^{-\lambda_-} \left(\frac{\nu'_- + \lambda'_-}{2} \right) = \xi_2 \quad . \quad (33)$$

Using (19), which relates ν'_- to λ'_- and ξ_0, ξ_1 , the last equation converts to

$$e^{-\lambda_-} \left[1 + \frac{R_-\nu'_-}{2} - \frac{R_-\lambda'_-}{2} \right] - 1 = \xi_2 \quad . \quad (34)$$

When $\xi_0 = \xi_1$, then $\nu'_- = -\lambda'_-$ and the old result of ²² is obtained.

Substituting again (19) and its derivative, given by (29), into the left hand side of (32) and reordering terms leads to

$$\begin{aligned} & - \left(\lambda''_- - \lambda'^2_- + \frac{2\lambda'_-}{R_-} \right) \\ & + \frac{1}{2} (\xi_0 - \xi_1) + \frac{1}{2} R_- (\xi'_0 - \xi'_1) - \frac{3}{4} \lambda'_- R_- (\xi_0 - \xi_1) + \frac{1}{8} R_-^2 (\xi_0 - \xi_1)^2 \\ & = \xi_1 \end{aligned} \quad . \quad (35)$$

In the first line we have $-\left(\lambda''_- - \lambda'^2_- + \frac{2\lambda'_-}{R_-}\right) = \frac{e^{\lambda_-}}{R_-} (R_- e^{-\lambda_-})''$, which can be directly verified by executing the second derivative. Shifting $[R_- e^{-\lambda_-}]'$ in (33) to one side and substituting this into $\frac{e^{\lambda_-}}{R_-} (R_- e^{-\lambda_-})''$, we arrive at

$$\begin{aligned} - \left(\lambda''_- - \lambda'^2_- + \frac{2\lambda'_-}{R_-} \right) &= \frac{e^{\lambda_-}}{R_-} (R_- e^{-\lambda_-})'' \\ &= \frac{e^{\lambda_-}}{R_-} \left(1 + \xi_2 - \frac{R_-^2}{4} (\xi_0 - \xi_1) e^{-\lambda_-} \right)' \\ &= \frac{e^{\lambda_-}}{R_-} \xi'_2 - \frac{1}{2} (\xi_0 - \xi_1) - \frac{R_-}{4} (\xi'_0 - \xi'_1) + \frac{\lambda_- R_-}{4} (\xi_0 - \xi_1) \quad . \end{aligned} \quad (36)$$

This can be inserted in the first line in (35) to obtain

$$\frac{e^{\lambda_-}}{R_-} \xi'_2 + \frac{1}{4} R_- (\xi'_0 - \xi'_1) - \frac{1}{2} \lambda'_- R_- (\xi_0 - \xi_1) + \frac{1}{8} R_-^2 (\xi_0 - \xi_1)^2 = \xi_1 \quad . \quad (37)$$

By also expressing the ξ_1 on the right hand side in (37) by ξ_2 using (28), we obtain finally

$$\begin{aligned} & \frac{e^{\lambda_-}}{R_-} \xi'_2 - e^{\lambda_-} \frac{2\xi_2}{R_-^2} \\ & = \\ & - \frac{1}{4} R_- (\xi'_0 - \xi'_1) + \frac{1}{2} \lambda'_- R_- (\xi_0 - \xi_1) - \frac{1}{8} R_-^2 (\xi_0 - \xi_1)^2 \quad . \end{aligned} \quad (38)$$

This is a differential equation relating ξ_2 with ξ_0 .

Using the derivative of $R_- e^{-\lambda_-}$ as given in (33), integrating it and setting the integration constant equal to $-2\mathcal{M}_-$, we obtain

$$R_- e^{-\lambda_-} = R_- - 2\mathcal{M}_- + \int \xi_2 dR_- - \frac{1}{4} \int e^{-\lambda_-} R_-^2 (\xi_0 - \xi_1) dR_- \quad . \quad (39)$$

The sum of the terms within the integral is nothing but Ξ_0 , as given in (23).

Now, we can use the connection of the ξ_μ functions in terms of the density ρ and pressure p , given in (27), with the result

$$R_- e^{-\lambda_-} = R_- - 2\mathcal{M}_- - \frac{8\pi\kappa}{c^2} \int R_-^2 \rho dR_- . \quad (40)$$

The same we would get by setting $\xi_0 = \xi_1$, i.e., $e^{-\lambda_-}$ depends only on the density.

However, $e^{-\nu_-}$, the metric component of $(DX_-^0)^2$, will not have such a simple expression. It is given by

$$e^{\nu_-} = e^{-\lambda_-} e^{\frac{1}{2} \int R_- (\xi_0 - \xi_1) dR_-} \\ e^{-\lambda_-} e^{\frac{8\pi\kappa}{c^2} \int R_- e^{\lambda_-} \left[\left(\frac{p}{c^2} \right) + \rho \right] dR_-} = e^{-\lambda_-} e^{f_-} . \quad (41)$$

The factor $e^{\frac{1}{2} \int R_- (\xi_0 - \xi_1) dR_-}$ is a positive function in R_- .

We now proceed in rewriting the expression in (40), considering first the last term:

$$- \frac{8\pi\kappa}{c^2} \int R_-^2 \rho dR_- = \frac{2\kappa}{c^2} M_{\text{de}}(R_-) . \quad (42)$$

$M_{\text{de}}(R_-)$ is the accumulated mass of the dark energy. The density will be proportional to a negative power of R_- , representing a decline of the dark energy density with increasing distance. Thus, the integral in R_- of the dark energy density will have a negative sign. This is accounted for on the left hand side of (42) by the multiplication with minus one, rendering the dark energy mass positive.

The metric element g_-^{11} is equal to $e^{-\lambda_-}$, which can be obtained from (40) and substituting into this equation the integral by (42). The expression for g_-^{11} is finally given by

$$e^{-\lambda_-} = 1 - \frac{2\mathcal{M}_-}{R_-} + \frac{1}{R_-} \frac{2\kappa}{c^2} M_{\text{de}}(R_-) \\ = 1 - \frac{2\mathcal{M}_-}{R_-} + \frac{2m_{\text{de}}(R_-)}{R_-} , \\ m_{\text{de}} = \frac{\kappa M_{\text{de}}(R_-)}{c^2} . \quad (43)$$

In (41) the $g_{00}^- = e^{\nu_-}$ was related to $e^{-\lambda_-}$, which is the g_-^{11} metric component. An additional factor appeared, abbreviated by e^{f_-} . With this and (43), the g_{00}^- component acquires the structure

$$g_{00}^- = \left(1 - \frac{2\mathcal{M}_-}{R_-} + \frac{2m_{\text{de}}}{R_-} \right) e^{f_-} , \quad (44)$$

where the function f_- has yet to be determined. In ²² the condition $g_{00} > 0$ after projection was imposed and the consequences are similar with respect to the redshift, the difference lying in the extra factor e^f , which is due to the assumption that in general $\xi_0 \neq \xi_1$.

Having resolved the components of the metric tensor, we will proceed to determine a relation between the pressure and the density of the energy outside the central mass.

In (27) the ξ_μ functions were related to the pressure, p , and the density, ρ . Substituting these relations into the differential equation (38), after a short rearrangement we arrive at

$$\frac{p'}{c^2} = \frac{1}{2}\lambda' \left(\frac{p}{c^2} + \rho \right) - \frac{8\pi\kappa}{c^2} \frac{1}{2} R_- e^{\lambda_-} \left(\frac{p}{c^2} + \rho \right)^2. \quad (45)$$

Obviously the derivative with respect to ρ does not appear!

Deriving $e^{-\lambda_-}$ in (43) with respect to R_- yields

$$-\lambda'_- e^{-\lambda_-} = \frac{2}{R_-^2} (\mathcal{M}_- - R_- m_{\text{de}}(R_-) + R_- m'_{\text{de}}(R_-)) \quad (46)$$

and we arrive at

$$\begin{aligned} p &= p(\rho) \\ m'_{\text{de}} &= -\frac{4\pi\kappa}{c^2} R_-^2 \rho \\ \frac{p'}{c^2} &= -\frac{\left(\frac{4\pi\kappa}{c^2} \frac{p}{c^2} R_-^3 + \mathcal{M}_- - m_{\text{de}}(R_-) \right)}{R_- (R_- - 2\mathcal{M}_- + 2m_{\text{de}}(R_-))} \left(\frac{p}{c^2} + \rho \right) \\ e^{\lambda_-} &= \left(1 - \frac{2\mathcal{M}_-}{R_-} + \frac{2m_{\text{de}}(R_-)}{R_-} \right) \\ \nu' &= 2 \frac{\left(\frac{4\pi\kappa}{c^2} \frac{p}{c^2} R_-^3 + \mathcal{M}_- - m_{\text{de}}(R_-) \right)}{R_- (R_- - 2\mathcal{M}_- + 2m_{\text{de}}(R_-))}, \end{aligned} \quad (47)$$

where we have added the equations for $\nu(R_-)$, e^{λ_-} , the relation of ρ to the derivative of m_{de} with respect to R_- and a *still unknown* equation of state $p = p(\rho)$. This set of equations is equivalent to the one of (14.25) in ³³, which were obtained within a model for a relativistic star structure. The difference is in the substitution of $m(r)$ by $(\mathcal{M}_- - m_{\text{de}}(R_-))$. Note, that when m_{de} increases, for an outside observer the effective mass of the object decreases.

3.2. The *pc-Schwarzschild metric*

In the last sub-section we obtained an analytic solution for the σ_- component of the metric. The one for the σ_+ component is identical to the one derived by e.g. Adler et al. ³³. In the σ_- component the functions Ω_- and f_- appear, the last in the g_{00}^- metric component. In the σ_+ component, however, no such functions appear. In order to rewrite the σ_+ component in a form similar to the one in the σ_- component, we introduce the definitions $\Omega_+ = 0$ and $f_+ = 0$. With the help of this both components can be written as

$$(g_{\mu\nu}^{\pm}) = \begin{pmatrix} \left(1 - \frac{2\mathcal{M}_{\pm}}{R_{\pm}} + \frac{\Omega_{\pm}}{R_{\pm}}\right) e^{f_{\pm}} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\mathcal{M}_{\pm}}{R_{\pm}} + \frac{\Omega_{\pm}}{R_{\pm}}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R_{\pm}^2 & 0 \\ 0 & 0 & 0 & -R_{\pm}^2 \sin^2 \theta \end{pmatrix} \quad (48)$$

As can be seen, the metric tensor has the equivalent functional form in both the σ_- and σ_+ component. We have used the notation

$$\begin{aligned} R_{\pm} &= r \pm l\dot{r} \\ \mathcal{M} &= \mathcal{M}_+ \sigma_+ + \mathcal{M}_- \sigma_- \\ \mathcal{M}_{\pm} &= m \\ \Omega &= 2m_{\text{de}} \sigma_- = \Omega_+ \sigma_+ + \Omega_- \sigma_- \\ \Omega_+ &= 0 \quad , \quad \Omega_- = 2m_{\text{de}} \\ f_+ &= 0 \quad , \quad f_- \neq 0 \quad . \end{aligned} \quad (49)$$

The condition $\mathcal{M}_{\pm} = m$ comes from the requirement that the standard GR should be contained in the limit of large distances r , which will be verified further below. The pseudo-real elements of the parameters are

$$M_R = m \quad , \quad \Omega_R = \frac{1}{2} (\Omega_+ + \Omega_-) = \frac{\Omega_-}{2} = m_{\text{de}} \quad . \quad (50)$$

Because now the metric tensors in both σ -component have the same functional form, the total pseudo-complex metric can be written as $g_{\mu\nu}(\Omega, f, R) = g_{\mu\nu}^+(\Omega_+, f_+, R_+) \sigma_+ + g_{\mu\nu}^-(\Omega_-, f_-, R_-) \sigma_-$, which gives

$$(g_{\mu\nu}) = \begin{pmatrix} \left(1 - \frac{2\mathcal{M}}{R} + \frac{\Omega}{R}\right) e^f & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\mathcal{M}}{R} + \frac{\Omega}{R}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 \sin^2 \theta \end{pmatrix} \quad (51)$$

and the projected metric, following our prescription, is

$$(g_{\mu\nu}(r)) = \begin{pmatrix} \left(1 - \frac{2m}{r} + \frac{\Omega_-}{2r}\right) e^{\frac{f_-}{2}} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2m}{r} + \frac{\Omega_-}{2r}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix} \quad . \quad (52)$$

The length squared element is, therefore, given by (see (15))

$$d\omega^2 = \left(1 - \frac{2m}{r} + \frac{\Omega_-}{2r}\right) e^{\frac{f_-}{2}} (dx^0)^2 - \left(1 - \frac{2m}{r} + \frac{\Omega_-}{2r}\right)^{-1} (dr)^2 - r^2 ((d\theta)^2 + \sin^2\theta (d\phi)^2) \quad . \quad (53)$$

In ²² we imposed the condition $g_{00}(r) > 0$, so that the signature for the time stays the same. In Chapter 5 and 6 we will consider a correction of $\Omega_- = \frac{B}{r^2}$ (with $B > 0$) as a model. The condition $g_{00}(r) > 0$ then is

$$1 - \frac{2m}{r} + \frac{B}{2r^3} > 0 \quad . \quad (54)$$

To find a value for B which satisfies this condition for all $r > 0$ we will have a look at the extremal value of g_{00} . As we know from the limiting behaviour of g_{00} ($g_{00} \rightarrow 1$ for $r \rightarrow \infty$ and $g_{00} \rightarrow +\infty$ for $r \rightarrow 0$) its extremal value will be a minimum. A quick calculation gives $r = \left(\frac{3}{4} \frac{B}{m}\right)^{1/2}$ as the value of the minimum of g_{00} . Inserting this in (54) yields

$$\left(\frac{3}{4} \frac{B}{m}\right)^{3/2} - B > 0 \quad \Rightarrow \quad B > \frac{64}{27} m^3 \quad . \quad (55)$$

The conclusions in ²² for the redshift in a Schwarzschild solution remain the same, i.e., after an increase of the redshift a minimum is reached. Because the potential is proportional to the square root of $g_{00}(r)$ ³⁷, this indicates a minimum in the potential, which is repulsive for lower radial distances r . As a consequence, the collapse of a star is halted latest at the minimum and it can not contract to a singularity. The star probably still oscillates around this minimum, which should be eventually observable. Further investigations concerning this aspect are required.

4. The pseudo-complex Reissner-Nordström solution

In this section we present our findings concerning the pc-Reissner-Nordström solution. Details can be found in ³⁵.

Since we consider a central, charged mass at rest, the spherical symmetry is conserved and the line element squared is again given by (17). Furthermore, we can adopt the Einstein equation (20) after adding the energy-momentum tensor for the electromagnetic field and the related conditions (21), (22) remain valid. Hence the Einstein equation is

$$\mathcal{R}_\mu^\nu - \frac{1}{2} g_\mu^\nu \mathcal{R} = \Xi_\mu^{\nu \text{ RN}} \sigma_- - \frac{8\pi\kappa}{c^2} T_\mu^{\nu \text{ em}} \quad , \quad (56)$$

where $T_\mu^{\nu \text{ em}}$ is given by ³³

$$T_{\mu}^{\nu \text{ em}} = \frac{\epsilon^2}{2c^2 r^4} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} . \quad (57)$$

Thereby ϵ depends on the charge Q in the following way

$$\epsilon = \frac{Q}{4\pi\epsilon_0} . \quad (58)$$

It is not certain, whether $\Xi_{\mu}^{\nu RN}$ is the same as in the Schwarzschild case, because there might be a coupling of the “dark” energy with the charge of the central mass. However at the moment we consider it as improbable that $\Xi_{\mu}^{\nu RN}$ is equal to Ξ_{μ}^{ν} , since the combination with the ideal fluid ansatz leads to an unphysical result (for detailed calculations the reader is referred to the appendix). Therefore, so far, no plausible approach exists for connecting the Reissner-Nordström source with the Schwarzschild one.

Nevertheless ξ_{μ}^{RN} can be defined such that a relation similar to (23) keeps valid, i.e.

$$\begin{aligned} -\frac{1}{4}e^{-\lambda_{RN}-\xi_0}\xi_0^{RN} + \frac{1}{4}e^{-\lambda_{RN}-\xi_1}\xi_1^{RN} + \frac{\xi_2^{RN}}{R_-^2} &= \Xi_0^{RN} \\ \frac{1}{4}e^{-\lambda_{RN}-\xi_0} - \frac{1}{4}e^{-\lambda_{RN}-\xi_1} + \frac{\xi_2^{RN}}{R_-^2} &= \Xi_1^{RN} \\ \frac{1}{4}e^{-\lambda_{RN}-}(\xi_0^{RN} + \xi_1^{RN}) &= \Xi_2^{RN} . \end{aligned} \quad (59)$$

4.1. Solving the Einstein equation

As in the previous chapter the σ_+ component does not differ from the usual GR field equations, which can be obtained in the same way as done in ³³. Thus we only have to solve for the σ_- component. In other words we are justified to restrict to the σ_- component only. We take into account, that the energy-momentum tensor of the electromagnetic contribution, $T_{\mu}^{\nu \text{ em}}$, is real, i.e., $T_{\mu}^{\nu \text{ em}} = T_{\mu}^{\nu \text{ em}}(\sigma_- + \sigma_+)$. Thus the σ_+ component is the same as the σ_- component. We use that $(\sigma_- + \sigma_+) = 1$. In addition we observe that

$$\mathcal{R} = \mathcal{R}_0^0 + \mathcal{R}_1^1 + \mathcal{R}_2^2 + \mathcal{R}_3^3 . \quad (60)$$

With this and $g_{\mu}^{\nu} = \delta_{\mu}^{\nu}$, the Einstein equations (56) for the σ_- component are

$$\begin{aligned}
\frac{1}{2}(\mathcal{R}_{-0}^0 - \mathcal{R}_{-1}^1 - \mathcal{R}_{-2}^2 - \mathcal{R}_{-3}^3) &= \Xi_0^{RN} - \frac{8\pi\kappa}{c^2} T_0^{0\ em} \\
\frac{1}{2}(\mathcal{R}_{-1}^1 - \mathcal{R}_{-0}^0 - \mathcal{R}_{-2}^2 - \mathcal{R}_{-3}^3) &= \Xi_1^{RN} - \frac{8\pi\kappa}{c^2} T_1^{1\ em} \\
\frac{1}{2}(\mathcal{R}_{-2}^2 - \mathcal{R}_{-0}^0 - \mathcal{R}_{-1}^1 - \mathcal{R}_{-3}^3) &= \Xi_2^{RN} - \frac{8\pi\kappa}{c^2} T_2^{2\ em} \\
\frac{1}{2}(\mathcal{R}_{-3}^3 - \mathcal{R}_{-0}^0 - \mathcal{R}_{-1}^1 - \mathcal{R}_{-2}^2) &= \Xi_3^{RN} - \frac{8\pi\kappa}{c^2} T_3^{3\ em} .
\end{aligned} \tag{61}$$

Taking the difference between the second and the third equation yields

$$\mathcal{R}_{-2}^2 - \mathcal{R}_{-3}^3 = \Xi_2^{RN} - \Xi_3^{RN} - \frac{8\pi\kappa}{c^2} (T_2^{2\ em} - T_3^{3\ em}) \tag{62}$$

and since the spherical symmetry and equation (57) demand $\mathcal{R}_{-2}^2 - \mathcal{R}_{-3}^3 = \frac{8\pi\kappa}{c^2} (T_2^{2\ em} - T_3^{3\ em}) = 0$, we obtain (note, that the lower index "-" refers to the σ_- component and *not* to the sign of a number)

$$\Xi_2^{RN} = \Xi_3^{RN} . \tag{63}$$

The difference of the zeroth and first equation leads to

$$\lambda'_{RN-} + \nu'_{RN-} = R_- e^{\lambda_{RN-}} (\Xi_1^{RN} - \Xi_0^{RN}) = \frac{R_-}{2} (\xi_0^{RN} - \xi_1^{RN}) . \tag{64}$$

After differentiation we get

$$\nu''_{RN-} = -\lambda''_{RN-} + \frac{1}{2} (\xi_0^{RN} - \xi_1^{RN}) + \frac{R_-}{2} (\xi_0^{RN'} - \xi_1^{RN'}) , \tag{65}$$

which is similar to equation (29).

By adding two times the second equation of (61) to the last difference and multiplying with $2e^{\lambda_{RN-}}$ we obtain

$$\nu''_{RN-} - \frac{\lambda'_{RN-} \nu'_{RN-}}{2} + \frac{\nu_{RN-}^2}{2} - \frac{2\lambda'_{RN-}}{R_-} = \xi_1^{RN} - \frac{2A}{R_-^4} e^{\lambda_{RN-}} , \tag{66}$$

where we used the abbreviation (see (58) for the definition of ϵ)

$$A := -\frac{4\pi\kappa\epsilon^2}{c^4} . \tag{67}$$

Moreover, after including (64) the sum of the equations with the index 0 and 1 in (61) leads to

$$(R_- e^{-\lambda_{RN-}})' = 1 + \xi_2^{RN} - \frac{1}{4} R_-^2 e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) + \frac{A}{R_-^2} , \tag{68}$$

which gives after an integration

$$e^{-\lambda_{RN-}} = 1 - \frac{2M_-}{R_-} + \frac{1}{R_-} \int \xi_2^{RN} dR_- + \frac{1}{4R_-} \int e^{-\lambda_{RN-}} R_-^2 (\xi_0^{RN} - \xi_1^{RN}) dR_- - \frac{A}{R_-^2} . \quad (69)$$

Now we can generate the Reissner Nordström equivalent of (38) through combining (64), (65) and (68) with (66) (a detailed calculation can be found in appendix (133))

$$\begin{aligned} & \frac{e^{\lambda_{RN-}}}{R_-} \xi_2^{RN'} - \xi_1^{RN} \\ &= \\ & -\frac{1}{4} R_- (\xi_0^{RN'} - \xi_1^{RN'}) + \frac{1}{2} \lambda_{RN-} R_- (\xi_0^{RN} - \xi_1^{RN}) - \frac{1}{8} R_-^2 (\xi_0^{RN} - \xi_1^{RN})^2 . \end{aligned} \quad (70)$$

This differential equation relates ξ_0^{RN} , ξ_1^{RN} and ξ_2^{RN} , whereby ξ_1^{RN} can again be expressed through ξ_2^{RN} , if the ideal fluid ansatz is used.

Through the use of (64) and (69) we can calculate the g_{00}^- -component of the metric

$$g_{00}^- = e^{\nu_{RN-}} = e^{-\lambda_{RN-}} e^{\frac{1}{2} \int R_- (\xi_0^{RN} - \xi_1^{RN}) dR_-} . \quad (71)$$

Within the ideal fluid ansatz the metric terms can be written in a form similar as in chapter 3

$$g_{11}^- = \left(1 - \frac{2M_-}{R_-} + \frac{2m_{de}^{RN}(R_-)}{R_-} - \frac{A}{R_-^2} \right)^{-1} \quad (72)$$

$$g_{00}^- = \left(1 - \frac{2M_-}{R_-} + \frac{2m_{de}^{RN}(R_-)}{R_-} - \frac{A}{R_-^2} \right) e^{f_{A-}} , \quad (73)$$

where, equivalent to (44), the function $f_{A-} = \frac{1}{2} \int R_- (\xi_0^{RN} - \xi_1^{RN}) dR_-$ has still to be determined.

4.2. The *pc-Reissner-Nordström* solution

After these calculations within the last sub-section 4.1 we now know the analytic solution of both components for the Reissner-Nordström problem. Hence we are now able to determine the real metric. We use the same notations as in 3.2 and additionally define $f_{A+} = 0$ (remember that we introduced this definition in order to have the same functional form of the metric tensor in both σ -components) and $f_A = f_{A+}\sigma_+ + f_{A-}\sigma_-$. Thus the complete pseudo-complex metric is

$$(g_{\mu\nu}) = \begin{pmatrix} \left(1 - \frac{2\mathcal{M}}{R} + \frac{\Omega_{RN}}{R} - \frac{A}{R^2}\right) e^{f_A} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2\mathcal{M}}{R} + \frac{\Omega_{RN}}{R} - \frac{A}{R^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -R^2 & 0 \\ 0 & 0 & 0 & -R^2 \sin^2 \theta \end{pmatrix} \quad (74)$$

and the projected metric is given by

$$\begin{pmatrix} \left(1 - \frac{2m}{r} + \frac{\Omega_{RN}}{2r} - \frac{A}{r^2}\right) e^{\frac{f_A}{2}} & 0 & 0 & 0 \\ 0 & -\left(1 - \frac{2m}{r} + \frac{\Omega_{RN}}{2r} - \frac{A}{r^2}\right)^{-1} & 0 & 0 \\ 0 & 0 & -r^2 & 0 \\ 0 & 0 & 0 & -r^2 \sin^2 \theta \end{pmatrix}. \quad (75)$$

With that we obtain the length element squared, which is

$$\begin{aligned} d\omega^2 = & \left(1 - \frac{2m}{r} + \frac{\Omega_{RN}}{2r} - \frac{A}{r^2}\right) e^{\frac{f_A}{2}} (dx^0)^2 \\ & - \left(1 - \frac{2m}{r} + \frac{\Omega_{RN}}{2r} - \frac{A}{r^2}\right)^{-1} (dr)^2 - r^2 ((d\theta)^2 + \sin^2 \theta (d\phi)^2). \end{aligned} \quad (76)$$

Thus, both g_{00} and g_{11} do not just get a charge dependence added as in GR, but the correction term is changed as well. Furthermore all terms of g_{00} are multiplied with a charge dependent factor $e^{\frac{f_A}{2}}$. In conclusion the metric components of the Reissner-Nordström metric are not the sum of the respective components of the Schwarzschild metric and the simple GR charge term anymore. Obviously, we do predict stronger deviations to GR as a priori expected. Also note, that according to (67) A is always negative, so that the charge prevents the collapse to a singularity, like in standard GR.

5. The pseudo-complex Kerr solution

In this section we will discuss our findings concerning a pseudo-complex Kerr solution. Some intermediate steps of the calculations can be found in the appendix of 36.

As previously mentioned, it is not that easy to find a pseudo-complex Kerr solution. To do so, we will follow an ansatz first made by Carter^{38,39}. He demanded the Klein-Gordon-Equation

$$\frac{1}{\Psi} \frac{\partial}{\partial x^\alpha} \left(\sqrt{-g} g^{\alpha\beta} \frac{\partial \Psi}{\partial x^\beta} \right) - m_0^2 \sqrt{-g} = 0 \quad (77)$$

to be separable. This yields an ansatz for the metric tensor

$$g_{\mu\nu} = \frac{1}{Z} \begin{pmatrix} \Delta_r C_\mu^2 - \Delta_\mu C_r^2 & 0 & 0 & \Delta_\mu C_r Z_r - \Delta_r C_\mu Z_\mu \\ 0 & -\frac{Z^2}{\Delta_r} & 0 & 0 \\ 0 & 0 & -\frac{Z^2}{\Delta_\mu} & 0 \\ \Delta_\mu C_r Z_r - \Delta_r C_\mu Z_\mu & 0 & 0 & \Delta_r Z_\mu^2 - \Delta_\mu Z_r^2 \end{pmatrix}, \quad (78)$$

where Δ_r and Z_r are functions of the variable r while Δ_μ and Z_μ are functions of $\mu = \cos \theta$. C_μ and C_r are constant factors, which will be determined later. The function Z is given as $Z = Z_r C_\mu - Z_\mu C_r$. Although the metric still shows a high symmetry, it is rather tedious to compute the Einstein equation. Some time can be saved by using a method from differential geometry, first introduced by Cartan⁴⁰, to obtain the Einstein tensor. For a full explanation of this method the reader is referred to^{37,41,42}. The Einstein tensor (remember that the Einstein equation is given by $G_\nu^\mu = -\frac{8\pi\kappa}{c^2} T_\nu^\mu$, with $G_\nu^\mu = \mathcal{R}_\nu^\mu - \frac{1}{2} g_\nu^\mu \mathcal{R}$) obtained is given by

$$\begin{aligned} G^0_0 &= \frac{1}{2Z} \Delta_{\mu|\mu\mu} + \frac{1}{Z^2} \Delta_r Z_{r|rr} + \frac{a^2}{4Z^3} \Delta_\mu (Z_{\mu|\mu}^2 + Z_{r|r}^2) \\ &\quad - \frac{3}{4Z^3} \Delta_r (Z_{\mu|\mu}^2 + Z_{r|r}^2) + \frac{a}{2Z^2} \Delta_{\mu|\mu} Z_{\mu|\mu} + \frac{1}{2Z^2} \Delta_{r|r} Z_{r|r} \\ G^0_3 &= -\frac{1}{2Z^2} \sqrt{\Delta_r \Delta_\mu} (a Z_{r|rr} + Z_{\mu|\mu\mu}) \\ G^1_1 &= \frac{1}{2Z} \Delta_{\mu|\mu\mu} + \frac{a^2}{4Z^3} \Delta_\mu (Z_{\mu|\mu}^2 + Z_{r|r}^2) - \frac{1}{4Z^3} \Delta_r (Z_{\mu|\mu}^2 + Z_{r|r}^2) \\ &\quad + \frac{a}{2Z^2} \Delta_{\mu|\mu} Z_{\mu|\mu} + \frac{1}{2Z^2} \Delta_{r|r} Z_{r|r} \\ G^2_2 &= \frac{1}{2Z} \Delta_{r|rr} - \frac{a^2}{4Z^3} \Delta_\mu (Z_{\mu|\mu}^2 + Z_{r|r}^2) + \frac{1}{4Z^3} \Delta_r (Z_{\mu|\mu}^2 + Z_{r|r}^2) \\ &\quad - \frac{a}{2Z^2} \Delta_{\mu|\mu} Z_{\mu|\mu} - \frac{1}{2Z^2} \Delta_{r|r} Z_{r|r} \\ G^3_3 &= \frac{1}{2Z} \Delta_{r|rr} - \frac{a}{Z^2} \Delta_\mu Z_{\mu|\mu\mu} - \frac{3a^2}{4Z^3} \Delta_\mu (Z_{\mu|\mu}^2 + Z_{r|r}^2) \\ &\quad + \frac{1}{4Z^3} \Delta_r (Z_{\mu|\mu}^2 + Z_{r|r}^2) - \frac{a}{2Z^2} \Delta_{\mu|\mu} Z_{\mu|\mu} - \frac{1}{2Z^2} \Delta_{r|r} Z_{r|r}, \end{aligned} \quad (79)$$

where the constant factors have been chosen as $C_r = a$, $C_\mu = 1$ ^{38,39,43} and the subscript $_{|\mu,r}$ stands for the derivative with respect to μ, r respectively. As before, all calculations stay the same, when we switch to a pseudo-complex description of the theory. Only the variational principle changes. Thus only the σ_- component of the equation needs to be considered, as the σ_+ part of it is identical to the classical Einstein equation. To solve now the Einstein equation $G_\mu^\nu = \Xi_\mu^\nu$, we will consider

similar combinations of (79) as Plebánski and Krasiński³⁹ did

$$\begin{aligned}
-\frac{1}{2Z^2}\sqrt{\Delta_{R-}\Delta_{\mu-}}(a_-Z_{R-|R-R-} + Z_{\mu-|\mu-\mu-}) &= \Xi^0_3 \\
\frac{1}{2Z}(\Delta_{\mu-|\mu-\mu-} + \Delta_{R-|R-R-}) &= \Xi^1_1 + \Xi^2_2 \\
\frac{a_-}{Z^2}\Delta_{\mu-}Z_{\mu-|\mu-\mu-} + \frac{a_-^2}{2Z^3}\Delta_{\mu-}(Z_{\mu-|\mu-}^2 + Z_{R-|R-}^2) &= \Xi^2_2 - \Xi^3_3 \\
\frac{1}{Z^2}\Delta_{R-}Z_{R-|R-R-} - \frac{1}{2Z^3}\Delta_{R-}(Z_{\mu-|\mu-}^2 + Z_{R-|R-}^2) &= \Xi^0_0 - \Xi^1_1 \\
\frac{1}{2Z}\Delta_{R-|R-R-} - \frac{a_-^2}{4Z^3}\Delta_{\mu-}(Z_{\mu-|\mu-}^2 + Z_{R-|R-}^2) - \frac{a_-}{2Z^2}\Delta_{\mu-|\mu-}Z_{\mu-|\mu-} \\
+ \frac{1}{4Z^3}\Delta_{R-}(Z_{\mu-|\mu-}^2 + Z_{R-|R-}^2) - \frac{1}{2Z^2}\Delta_{R-|R-}Z_{R-|R-} &= \Xi^2_2 \quad . \quad (80)
\end{aligned}$$

We will treat the Ξ_μ^ν as arbitrary functions in R_- and μ_- at first. This allows us to choose them properly, so that the equations (80) can be solved. The first step consists in setting $\Xi_0^3 = 0$ and thus the first line in (80) becomes

$$aZ_{R-|R-R-} + Z_{\mu-|\mu-\mu-} = 0 \quad . \quad (81)$$

Choosing $\Xi_0^3 \neq 0$ would not allow an analytic solution, i.e., the assumption $\Xi_0^3 = 0$ is for convenience. (81) is formally identical to the classical case³⁹. We have a sum of two functions of different variables equal to a constant. Thus both have to be constant and we can conclude

$$Z_{R-} = CR_-^2 + C_1R_- + C_2 \quad \text{and} \quad Z_{\mu-} = -a_-C\mu_-^2 + C_3\mu_- + C_4 \quad . \quad (82)$$

The next ad hoc choice is made with $\Xi_2^2 = \Xi_3^3$ and with the third equation in (80) we arrive, after some manipulations, at

$$C_4 = \frac{C_2}{a_-} - \frac{C_1^2 + C_3^2}{4a_-C} \quad . \quad (83)$$

Inserted in (81) we can observe, that the transformation $\mu_- = \mu'_- + \frac{C_3}{2a_-C}$ together with a redefinition $C_2 = a_-C'_2 + \frac{C_1^2}{4C}$ has the same effect as if we choose $C_3 = 0$ ³⁹. Thus we are left with

$$Z_{R-} = C \left(R_- + \frac{C_1}{2C} \right)^2 + a_-C'_2 \quad , \quad Z_{\mu-} = -aC\mu_-^2 + C'_2 \quad . \quad (84)$$

Another transformation, now for the variable R_- , yields the same as if we set $C_1 = 0$. As the factor $Z = Z_{R-} - a_-Z_{\mu-}$ is independent of C'_2 we can choose $C'_2 = a_-C$ just as in the classical case³⁹. The final step here consist of setting $C = 1$ by redefining $\Delta_{\mu-}$ and Δ_{R-} . Therefore, we have determined the functions

$$Z_{R-} = R_-^2 + a_-^2 \quad , \quad Z_{\mu-} = a_-(1 - \mu_-^2) \quad , \quad (85)$$

which again are formally identical to the classical solution^{38,39}.

Now, we consider the second equation of (80) with the assumption $\Xi_1^1 + \Xi_2^2 = \frac{1}{2Z} \sum_{n=3}^{\infty} \frac{\tilde{B}_n}{R_-^n}$ (the right hand side simulates the contribution of $T_{de}^{\mu\nu}$ of the "dark energy"), which yields^a

$$\Delta_{R_-|R_-R_-} + \Delta_{\mu_-|\mu_- \mu_-} - \sum_{n=3}^{\infty} \frac{\tilde{B}_n}{R_-^n} = 0 \quad . \quad (86)$$

Again we have two functions of different variables to be equal. This leads to

$$\begin{aligned} \Delta_{R_-} &= ER_-^2 - 2\mathcal{M}_-R_- + E_2 + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}} \quad , \\ \Delta_{\mu_-} &= -E\mu_-^2 + E_3\mu_- + E_4 \quad . \end{aligned} \quad (87)$$

Inserting this and (85) into the last equation of (80) we get after some algebra

$$\sum_{n=3}^{\infty} \left(\frac{\tilde{B}_n}{R_-^{n-2}} \left(\frac{1}{n-2} + \frac{1}{2} \right) + \frac{\tilde{B}_n a^2 \mu_-^2}{2R_-^n} \right) + (E_2 - E_4 a^2) = Z^2 \Xi_2^2 \quad . \quad (88)$$

If we now chose

$$\Xi_2^2 = \frac{1}{Z^2} \sum_{n=3}^{\infty} \left(\frac{\tilde{B}_n}{R_-^{n-2}} \left(\frac{1}{n-2} + \frac{1}{2} \right) + \frac{\tilde{B}_n a^2 \mu_-^2}{2R_-^n} \right) \quad , \quad (89)$$

the previous equation can be fulfilled while retaining the condition $(E_2 - E_4 a^2) = 0$ as in the classical case.

In order to determine the remaining constants in Δ_{R_-} and Δ_{μ_-} we will proceed analogously to Plebanski and Krasiński³⁹. At first we set $E_3 = 0$, otherwise one would have a term proportional to $\mu_- = \cos \theta_-$, which violates the symmetry with respect to a reflection on the equatorial plane. To avoid a coordinate singularity at the poles, we set $E = 1$. Finally we choose $E_4 = 1$ to get the correct Schwarzschild metric in the limit $a_- \rightarrow 0$. This leaves us then with

$$\begin{aligned} Z_{R_-} &= R_-^2 + a_-^2 \quad , \quad Z_{\mu_-} = a_-(1 - \mu_-^2) \quad , \\ \Delta_{R_-} &= R_-^2 - 2\mathcal{M}_-R_- + a_-^2 + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}} \quad , \\ \Delta_{\mu_-} &= 1 - \mu_-^2 \quad , \quad Z = Z_{R_-} - a_- Z_{\mu_-} = R_-^2 + a_-^2 \mu_-^2 \quad . \end{aligned} \quad (90)$$

This can be inserted into (78) and, together with $\mu = \cos \theta$, we get the σ_- -part of

^aInstead of the whole series one could choose only one or several of the terms in the series. Later we will restrict our discussions to the case $n = 3$ as in the previous sections.

the metric

$$\begin{aligned}
g_{00}^- &= \frac{R_-^2 - 2\mathcal{M}_- R_- + a_-^2 \cos^2 \theta_- + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}}}{R_-^2 + a_-^2 \cos^2 \theta_-} \\
g_{11}^- &= -\frac{R_-^2 + a_-^2 \cos^2 \theta_-}{R_-^2 - 2\mathcal{M}_- R_- + a_-^2 + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}}} \\
g_{22}^- &= -R_-^2 - a_-^2 \cos^2 \theta_- \\
g_{33}^- &= -(R_-^2 + a_-^2) \sin^2 \theta_- - \frac{a_-^2 \sin^4 \theta_- \left(2\mathcal{M}_- R_- - \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}} \right)}{R_-^2 + a_-^2 \cos^2 \theta_-} \\
g_{03}^- &= \frac{-a_- \sin^2 \theta_- 2\mathcal{M}_- R_- + a_- \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{R_-^{n-2}} \sin^2 \theta_-}{R_-^2 + a_-^2 \cos^2 \theta_-} . \tag{91}
\end{aligned}$$

Note, that in spite of all assumptions made, (91) represents a new Kerr solution also in standard GR with a special $T^{\mu\nu}$ tensor. This is because, as mentioned before, each σ_{\pm} component describes one particular GR with a given symmetric metric.

The σ_+ -component matches the classical Kerr solution. Finally, projecting the pc-metric on its real part, as described above, yields the metric

$$\begin{aligned}
g_{00}^{\text{Re}} &= \frac{r^2 - 2mr + a^2 \cos^2 \theta + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{2r^{n-2}}}{r^2 + a^2 \cos^2 \theta} \\
g_{11}^{\text{Re}} &= -\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2 + \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{2r^{n-2}}} \\
g_{22}^{\text{Re}} &= -r^2 - a^2 \cos^2 \theta \\
g_{33}^{\text{Re}} &= -(r^2 + a^2) \sin^2 \theta - \frac{a^2 \sin^4 \theta \left(2mr - \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{2r^{n-2}} \right)}{r^2 + a^2 \cos^2 \theta} \\
g_{03}^{\text{Re}} &= \frac{-a \sin^2 \theta 2mr + a \sum_{n=3}^{\infty} \frac{1}{(n-1)(n-2)} \frac{\tilde{B}_n}{2r^{n-2}} \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} . \tag{92}
\end{aligned}$$

For the further discussions we will only consider the case $n = 3$, which results

in the metric

$$\begin{aligned}
g_{00}^{\text{Re}} &= \frac{r^2 - 2mr + a^2 \cos^2 \theta + \frac{B}{2r}}{r^2 + a^2 \cos^2 \theta} \\
g_{11}^{\text{Re}} &= -\frac{r^2 + a^2 \cos^2 \theta}{r^2 - 2mr + a^2 + \frac{B}{2r}} \\
g_{22}^{\text{Re}} &= -r^2 - a^2 \cos^2 \theta \\
g_{33}^{\text{Re}} &= -(r^2 + a^2) \sin^2 \theta - \frac{a^2 \sin^4 \theta (2mr - \frac{B}{2r})}{r^2 + a^2 \cos^2 \theta} \\
g_{03}^{\text{Re}} &= \frac{-a \sin^2 \theta (2mr + a \frac{B}{2r}) \sin^2 \theta}{r^2 + a^2 \cos^2 \theta} .
\end{aligned} \tag{93}$$

As this metric (93) represents the pseudo-complex equivalent to the Kerr solution, it is of interest, whether one can still identify the parameter a with the angular momentum J . To see that this identification still holds, we will follow Adler et al.³³ and expand the line element given by (93) linear in a

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2m}{r} + \frac{B}{2r^3}\right) dt^2 - \frac{1}{1 - \frac{2m}{r} + \frac{B}{2r^3}} dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \\
&\quad + 2a \sin^2 \theta \left(-\frac{2m}{r} + \frac{B}{2r^3}\right) d\phi dt .
\end{aligned} \tag{94}$$

This expansion represents the limit of a slowly rotating body. Next we expand (94) linear in $\frac{1}{r}$. This is the limit for large distances. The line element now takes the form

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2m}{r}\right) dt^2 - \left(1 + \frac{2m}{r}\right) dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \\
&\quad - 2a \sin^2 \theta \frac{2m}{r} d\phi dt .
\end{aligned} \tag{95}$$

In the following discussion we will focus on the term proportional to $d\phi dt$.

$$-2a \sin^2 \theta \frac{2m}{r} d\phi dt . \tag{96}$$

To compare this term with the calculations of Lense and Thirring^{33,44}

$$\begin{aligned}
ds^2 &= \left(1 - \frac{2m}{\rho}\right) dt^2 - \left(1 + \frac{2m}{\rho}\right) [d\rho^2 + \rho^2 (d\theta^2 + \sin^2 \theta d\phi^2)] \\
&\quad + \frac{4\kappa J}{c^3 \rho} \sin^2 \theta d\phi dt ,
\end{aligned} \tag{97}$$

we note, that for large distances the two radial variables r and ρ (isotropic coordinates, used by Lense and Thirring) coincide. This finally yields the same connection between a and the angular momentum J as in the classical case

$$a = -\frac{\kappa J}{mc^3} . \tag{98}$$

Obviously, the parameter a can still be identified with the angular momentum of the source.

The classical Kerr solution shows some special hypersurfaces which are of great physical interest. One of these we already know from the classical Schwarzschild solution: In the orbital plane the radius of this sphere is at $r = 2m$, a sphere with infinite red shift. In ²² the specific choice $B > 2m^2$ had the effect, that this infinite red shift surface and the singularity at the origin vanished. For corrections proportional to $\frac{B}{2r^3}$ this happens for $B > (4/3m)^3$ (see (55)). We will now investigate the influence of the additional term proportional to B on the existence of an event horizon for the Kerr metric.

As shown in ³³ surfaces corresponding to $g_{00} = 0$ can be passed in both directions by an observer (except at the poles), e.g. these surfaces are no event horizons.

The property of a surface to be an event horizon can be investigated through the norm of its normal vector n_α . Only if it is negative, physical observers can pass in both directions. A normal vector with positive norm corresponds to a timelike surface. Such surfaces can only be passed in one direction by physical observers. Now we will look for time independent axially symmetric surfaces with a null normal vector. These surfaces can be described by ³³

$$u(r, \theta) = \text{const} \quad . \quad (99)$$

Their normal vector is then given by

$$n_\alpha = \left(0, \frac{\partial u}{\partial r}, \frac{\partial u}{\partial \theta}, 0 \right) \quad . \quad (100)$$

Setting the norm $n_\alpha n^\alpha = 0$ yields the equation

$$\left(r^2 - 2mr + a^2 + \frac{B}{2r} \right) \left(\frac{\partial u}{\partial r} \right)^2 + \left(\frac{\partial u}{\partial \theta} \right)^2 = 0 \quad , \quad (101)$$

which can be solved by a product ansatz $u = R(r)\Theta(\theta)$

$$- \left(r^2 - 2mr + a^2 + \frac{B}{2r} \right) \left(\frac{\partial R}{\partial r} \right)^2 = \left(\frac{\partial \Theta}{\partial \theta} \right)^2 \quad . \quad (102)$$

Both sides of this equation depend on different variables and thus have to be constant. In analogy to ³³ we will call that constant λ which then gives

$$\Theta = Ae^{\sqrt{\lambda}\theta} \quad . \quad (103)$$

This expression however is not periodic in θ and therefore can't describe a surface except for the case where $\lambda = 0$, which means $\Theta = \text{const}$. The remaining equation for R then is

$$\left(r^2 - 2mr + a^2 + \frac{B}{2r} \right) \left(\frac{\partial R}{\partial r} \right)^2 = 0 \quad . \quad (104)$$

Excluding the trivial case $\frac{\partial R}{\partial r} = 0$ we are left with the solution of

$$\left(r^2 - 2mr + a^2 + \frac{B}{2r}\right) = 0 \quad . \quad (105)$$

Possible physical solutions for r are given by positive real roots of the cubic polynomial

$$p(r) = r^3 - 2mr^2 + a^2r + \frac{B}{2} \quad . \quad (106)$$

Since the derivative

$$p'(r) = 3r^2 - 4mr + a^2 \quad (107)$$

is positive for all $r \leq 0$, from $p(0) = B/2 > 0$ and $\lim_{r \rightarrow -\infty} p(r) = -\infty$ it follows that $p(r)$ has always exactly one negative real root, which is not relevant for our argument. Depending on the parameters a^2 and B there might be two more real roots, which then have to be positive numbers and thus would represent possible solutions of (105). It is well known that a cubic function has three distinct real roots if it has a positive discriminant⁴⁵. For $p(r)$ the parameter dependent discriminant $D(a^2, B)$ reads

$$D(a^2, B) = \frac{1}{27} \left(4(4m^2 - 3a^2)^3 - \left(18ma^2 - 16m^3 + \frac{27}{2}B \right)^2 \right) \quad . \quad (108)$$

It is easy to see that a first condition for $D(a^2, B) > 0$ is already given by $a^2 < (4/3)m^2$. We rewrite the condition $D(a^2, B) > 0$ by use of the parametrization $a^2 = \epsilon(4/3)m^2$, with $\epsilon \in [0, 1]$, and obtain

$$4(4(1 - \epsilon)m^2)^3 > \left(8m^3(3\epsilon - 2) + \frac{27}{2}B \right)^2 \quad . \quad (109)$$

Now we determine the maximum parameter value B^* for which this condition can be satisfied. The left hand term monotonically decreases with increasing ϵ , whereas the right hand term monotonically increases as long as the term in the bracket is positive. If for some ϵ and B the condition is met with a negative term in the bracket on the right hand side, we can choose a larger B such that this term is positive and the condition is still fulfilled. It follows that the maximum value B^* satisfying the condition (109) is obtained for $\epsilon = 0$, or equivalently $a = 0$. In this case (109) reads

$$4(4m^2)^3 = (16m^3)^2 > \left(\frac{27}{2}B - 16m^3 \right)^2 \quad , \quad (110)$$

which yields $B^* = (4/3)^3 m^3$. This value corresponds to the limiting case for the Schwarzschild solution including corrections proportional to $\frac{B}{2r^3}$ as discussed at the end of chapter 3 (see (55)). We conclude that also for $a > 0$, for $B > B^*$ there are no positive real roots of (105) and therefore just as in the Schwarzschild case the modified Kerr solution shows no event horizons. Note that there also are no

surfaces of infinite redshift as $g_{00} = 0$ is included in our discussion of (105) because a^2 and $a^2 \cos^2 \theta$ have the same range of values.

6. Some experimental considerations

In this section we discuss the phenomenon, that a light emitting blob of plasma is circulating an *Active Galactic Nucleus* (AGN). These objects are observed and discussed in the literature^{46,47,48}. Orbiting around the AGN, near to the Schwarzschild radius, they appear at a well determined frequency. Using standard GR for the Kerr solution one obtains from the orbiting frequency the distance r of this plasma blob from the center. This assumes standard GR. The deduced r value will change, of course, when pc-GR is applied. Here, we will determine the frequency as a function of the radial distance for the case of a black hole in Einstein's GR, for the pc-Schwarzschild solution and the same for the Kerr solution. Differences will be pointed out, in particular such signals, which, eventually, can not be explained by standard GR, may serve as a possible sign for pc-GR.

In a first step, we consider a non-rotating gray star. The real situation would be a rotating gray star, which we will discuss further below. The Schwarzschild case serves for illustrating purposes.

Furthermore, we restrict to a simplified circular orbit. The Lagrange function in standard GR is given by^b

$$L = g_{00} c^2 \dot{t}^2 + g_{11} \dot{r}^2 - r^2 [\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2] = \frac{ds^2}{ds^2} = 1 \quad (111)$$

The dot indicates the derivative with respect to s . Using the variational principle for the geodesic equations, we arrive at the following equations of motion

$$\begin{aligned} \frac{d}{ds}(g_{00} c^2 \dot{t}) &= 0 \\ \frac{d}{ds}(2g_{11} \dot{r}) &= \frac{\partial g_{00}}{\partial r} c^2 \dot{t}^2 + \frac{\partial g_{11}}{\partial r} \dot{r}^2 - 2r[\dot{\theta}^2 + \sin^2 \theta \dot{\phi}^2] \\ \frac{d}{ds}(-2r^2 \dot{\theta}) + 2r^2 \sin \theta \cos \theta \dot{\phi}^2 &= 0 \\ \frac{d}{ds}(-2r^2 \sin^2 \theta \dot{\phi}) &= 0 \end{aligned} \quad (112)$$

The structure will not be much different, when we move to the pseudo-complex description. The change consists in modified expressions for the metric components.

In the Schwarzschild case, we can restrict to a circular motion with $\theta = \frac{\pi}{2}$. Then the former equations simplify to

^bWe neglect in the further discussions the factor $e^{\frac{f_-}{2}}$, assuming that it is small.

$$\begin{aligned}
\frac{d}{ds}(g_{00}c^2\dot{t}) &= 0 \\
\frac{d}{ds}(g_{11}\dot{r}) &= \frac{\partial g_{00}}{\partial r}c^2\dot{t}^2 + \frac{\partial g_{11}}{\partial r}\dot{r}^2 - 2r\dot{\phi}^2 \\
\frac{d}{ds}(-2r^2\dot{\phi}) &= 0
\end{aligned} \tag{113}$$

From the third equation follows the conservation of angular momentum

$$H = r^2\dot{\phi} = r^2\frac{d\phi}{dt}\dot{t} = r^2\omega\dot{t} = \text{const} \quad , \tag{114}$$

where H is proportional to the angular momentum.

From the first equation in (113) the derivative of the time with respect to the eigentime (the derivative is indicated by a dot) is determined, i.e.,

$$\begin{aligned}
2g_{00}c^2\dot{t} &= 2Ac \\
\dot{t} &= \frac{A}{g_{00}c}
\end{aligned} \tag{115}$$

Restricting to circular orbits, we have $r = r_0$ and $\dot{r} = 0$. Due to this, the integration constant can be related to the frequency ω of the circular motion. To do that we insert $\theta = \frac{\pi}{2}$, $\dot{\phi} = \omega\dot{t}$ and $\dot{r} = \dot{\theta} = 0$ into equation (111) which then becomes

$$g_{00}c^2\dot{t}^2 - r^2\dot{\phi}^2 = 1 \quad \Rightarrow \quad g_{00}c^2\dot{t}^2 - r^2\omega^2\dot{t}^2 = 1 \quad \Leftrightarrow \quad \dot{t}^2 = \frac{1}{g_{00}c^2 - r^2\omega^2} \quad . \tag{116}$$

Now we can use (115) and obtain

$$A = \frac{g_{00}c}{\sqrt{g_{00}c^2 - r^2\omega^2}} \quad . \tag{117}$$

The second equation in (113) simplifies to

$$0 = \frac{\partial g_{00}}{\partial r}c^2\dot{t}^2 - 2r\dot{\phi}^2 \quad . \tag{118}$$

Considering that $\dot{\phi} = \frac{d\phi}{ds} = \frac{d\phi}{dt}\frac{dt}{ds} = \omega\dot{t}$ we obtain

$$0 = \frac{\partial g_{00}c^2}{\partial r} - 2r\omega^2 \quad \Leftrightarrow \quad \omega = \pm c\sqrt{\frac{1}{2r}\frac{\partial g_{00}}{\partial r}} \quad . \tag{119}$$

This expression is valid for both the GR and for the pc-GR Schwarzschild case. Using the g_{00} component for standard GR and then for pc-GR (assuming a correction to the metric of $\frac{B}{2r^3}$, see the end of chapter 3.2), we obtain respectively

$$\begin{aligned}\omega_{\text{GR}} &= c\sqrt{\frac{m}{r^3}} \\ \omega_{\text{pc-GR}} &= \sqrt{\frac{m}{r^3} - \frac{3B}{4r^5}} \quad .\end{aligned}\tag{120}$$

These are two different relations, which yield for a fixed, observed frequency two different radial distances.

In the pc-GR there is also a *last stable orbit*. The reason for that is the structure of the effective potential, which analogous to Misner et al. (page 639) ³⁷ has the form

$$V^2 = \left(1 - \frac{2m}{r} + \frac{B}{2r^3}\right) \left[1 + \frac{L^2}{r^2}\right] \quad .\tag{121}$$

Choosing as $B = \frac{64}{27}m^3$, the first factor in (121) becomes zero at two thirds of the Schwarzschild radius (see Fig. 1). For this limit, the behavior for larger r is the same as in the standard theory.

A local minimum exists for $r > \frac{4m}{3}$ and vanishes at approximately $r = 5.25m$. This means that the last stable orbit is at a slightly smaller position than in standard GR, where it is at $r = 6m$, three times the Schwarzschild radius. Thus, one possibility is to look for orbits which exist at smaller distances than $6m$, if the gray star does not rotate.

There is another consequence: The maximal observable velocity of these blobs is given by ωr . In case of GR and pc-GR respectively the maximal velocity is given by

$$\begin{aligned}v_{\text{max}}^{\text{GR}} &= c\sqrt{\frac{m}{6m}} \lesssim 0.409c \\ v_{\text{max}}^{\text{pc-GR}} &= \pm c\sqrt{\frac{m}{r} - \frac{3B}{4r^3}} < c\sqrt{\frac{m}{r} - \frac{16m^3}{9r^3}} \\ &< 0.423c \quad ,\end{aligned}\tag{122}$$

where in the last relation we used for r the value at the last stable orbit at $5.25m$. Obviously in both cases the maximal velocities are nearly equal.

Of course, this is only valid for a non-rotating large central mass. In a real situation one has to study the Kerr solution, which we will consider next:

The calculations follow along the same lines as for the non-rotating large mass. The Lagrange function is given by

$$L = g_{00}c^2\dot{t}^2 + g_{11}c^2\dot{r}^2 + g_{22}\dot{\theta}^2 + g_{33}\dot{\phi}^2 + 2g_{03}c\dot{t}\dot{\phi} \quad .\tag{123}$$

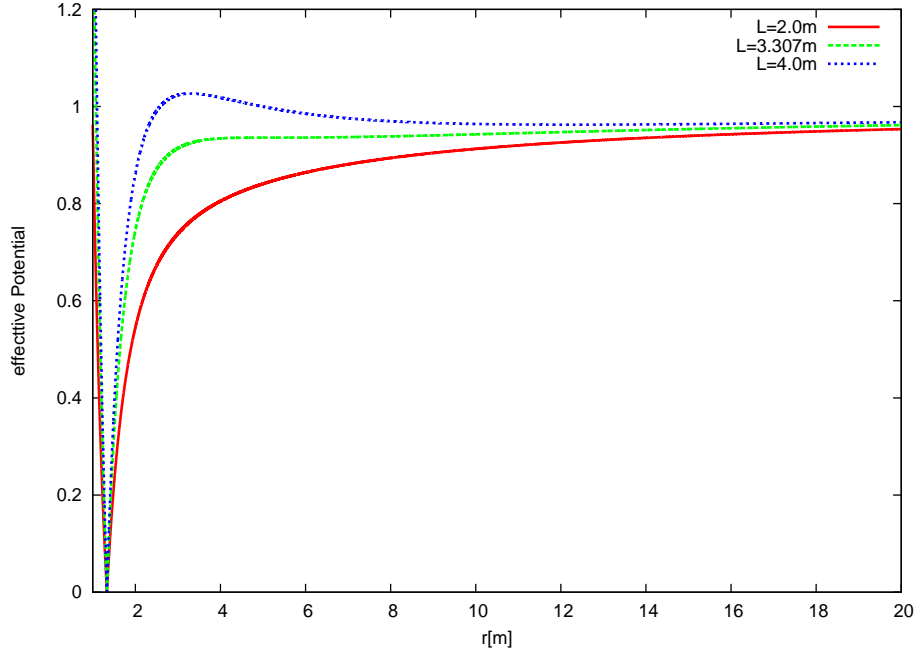


Fig. 1. The effective potential for the pc-Schwarzschild solution. The value $L = 3.307m$ corresponds to the last stable orbit (saddle point at $r \approx 5.25$).

As in the Schwarzschild case, the equation of motion for the radial part is used to determine the orbital frequency, i.e.,

$$\begin{aligned} \frac{d}{ds} (2g_{11}\dot{r}) &= \frac{\partial g_{00}}{\partial r} c^2 \dot{t}^2 + \frac{\partial g_{11}}{\partial r} \dot{r}^2 + \frac{\partial g_{22}}{\partial r} \dot{\theta}^2 + \frac{\partial g_{33}}{\partial r} \dot{\phi}^2 + 2 \frac{\partial g_{03}}{\partial r} c \dot{t} \dot{\phi} \\ &=: g'_{00} c^2 \dot{t}^2 + g'_{11} \dot{r}^2 + g'_{22} \dot{\theta}^2 + g'_{33} \dot{\phi}^2 + 2g'_{03} c \dot{t} \dot{\phi} \quad , \end{aligned} \quad (124)$$

where the prime indicates a partial derivative with respect to the radial distance. We restrict to orbital motion in the horizontal plane, with $r = r_0$, $\dot{r} = 0$, $\theta = \frac{\pi}{2}$ and $\dot{\theta} = 0$. With this, the last equation simplifies to

$$\begin{aligned} 0 &= g'_{00}(r_0) c^2 \dot{t}^2 + g'_{33}(r_0) \dot{\phi}^2 + 2g'_{03}(r_0) c \dot{t} \dot{\phi} \\ &= g'_{00}(r_0) c^2 \dot{t}^2 + g'_{33}(r_0) \omega^2 \dot{t}^2 + 2g'_{03}(r_0) \omega c \dot{t}^2 \quad . \end{aligned} \quad (125)$$

and the orbital frequency results as

$$\omega_{\pm}^{\text{pc-GR}} = -c \frac{g'_{03}}{g'_{33}} \pm c \sqrt{\left(\frac{g'_{03}}{g'_{33}} \right)^2 - \frac{g'_{00}}{g'_{33}}} \quad . \quad (126)$$

Substituting the metric components for the pc-Kerr-problem (see eqs. (93),(98))

$$\begin{aligned} g_{00} &= 1 - \frac{2m}{r} + \frac{B}{2r^3} \\ g_{03} &= \frac{-2amr + a\frac{B}{2r}}{r^2} = a \left(-\frac{2m}{r} + \frac{B}{2r^3} \right) \\ g_{33} &= -(r^2 + a^2) - a^2 \left(\frac{2m}{r} - \frac{B}{2r^3} \right) \quad , \end{aligned} \quad (127)$$

we obtain finally for the orbital frequency

$$\omega_{\pm}^{\text{pc-GR}} = \frac{ac \left(\frac{2m}{r^2} - \frac{3B}{2r^4} \right)}{2r - a^2 \left(\frac{2m}{r^2} - \frac{3B}{2r^4} \right)} \pm \frac{c \sqrt{\frac{4m}{r} - \frac{3B}{r^3}}}{\left| 2r - a^2 \left(\frac{2m}{r^2} - \frac{3B}{2r^4} \right) \right|} \quad . \quad (128)$$

The standard GR solution is obtained by setting $B = 0$. If, as usual, $a > 0$ is assumed, then the negative sign corresponds to an orbital motion in phase with the rotation of the gray star and the positive sign is for a motion opposite to its rotation^c.

In Fig. 2 the orbital frequency $\omega = 2\pi\nu$ is plotted versus r , for an $a = 0.995$. The units are given in $\frac{c}{m}$. Using the literature values for the gravitational constant ($6.674 \cdot 10^{-11} m^3 kg^{-1} s^{-2}$), the speed of light ($3 \cdot 10^8 m s^{-1}$) and the mass of the sun ($2 \cdot 10^{30} kg$), we obtain

$$\frac{c}{m} \approx 1.22 \times 10^7 \frac{M_{\text{sun}}}{M} \text{ min}^{-1} \quad . \quad (129)$$

The unit is in one over minutes, M_{sun} is the mass of the sun and M is the mass of the central object.

Taking into account that the mass of the object in Sagittarius A, the center of our galaxy, is of the order of $3 \times 10^7 M_{\text{sun}}$, we get

$$\frac{c}{m} \approx 3.3 \text{ min}^{-1} \quad . \quad (130)$$

ω is given by $2\pi\nu$, i.e., for ν we obtain, using the maximum value of $\omega \approx 0.219$, $\nu = 0.115 \text{ min}^{-1}$. This corresponds to the minimal time of 8.7 min for a mass to circulate around the large mass. Fig. 2 indicates that the standard GR produces larger frequencies, thus a lesser time to circulate the large mass, which should be measurable. The pc-Kerr solution exhibits a maximum frequency, which is due to the change of sign in the second derivative of g_{00} . This property implies a minimal time of orbit, which would not exist in standard GR. This should be a clear sign to distinguish between GR and pc-GR.

^cThe parameter a sometimes is defined with an opposite sign like in the book of Misner et al. ³⁷.

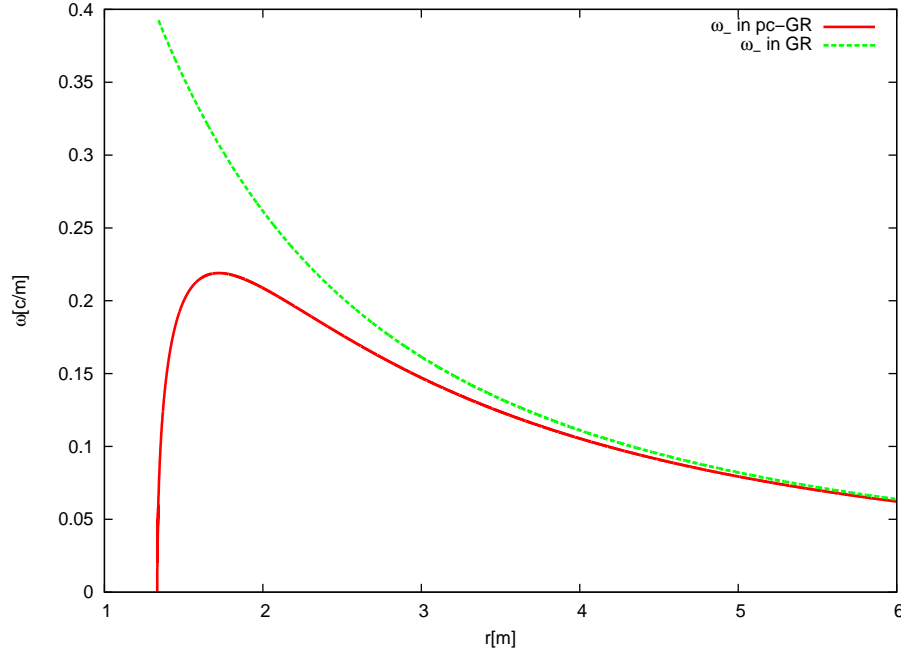


Fig. 2. The orbital frequency of a co-rotating object in a stable orbit versus the radial distance r for $a = 0.995$. The plot for ω in GR starts at the last stable orbit, which is $r = 1.341m$. For ω in pc-GR it starts at $r = \frac{4}{3}m$ - for radii below this value equation (128) has no real solutions anymore. Thus we do not expect circular geodesic orbits below this value of r . The m is given by $\frac{\kappa M}{c^2}$, with κ being the gravitational constant. Sometimes m is denoted as r_g in literature^{47,48}.

7. Conclusions

In this paper we revisited the pseudo-complex General Relativity, as proposed by P.O. Hess and W. Greiner²². Inconsistencies in the projection to real results were found and discussed. A corrected projection rule was presented.

The new procedure was applied to the case of a non-rotating gray star, the pseudo-complex Schwarzschild solution.

In the last two chapters new pseudo-complex solutions were constructed, namely the pseudo-complex Reissner-Nordström solution for a charged gray star and the pseudo-complex Kerr solution for a rotating gray star. The calculational procedures were rather complex, nevertheless analytic solutions were found.

In all cases, the modified variational principle introduced contributions, which can be interpreted as dark energy, acting repulsively such that the formation of an event horizon and a singularity at the center is avoided. This is a most important result, as any proper theory should not contain singularities.

The origin of the dark energy stems from different field equations (different

with respect to Einstein's GR). This again, is most satisfying: Dark energy can be introduced by modified field equations in the pseudo-complex treatment of general relativity.

In section 2 we also commented on the corrections to the theory, when the contributions of the minimal length scale are included and their consequences, resulting in the dispersion relation of a particle. As explained, because of the change in the four-velocity is considered to be small (small acceleration), the contribution of the minimal length can safely be neglected. Nevertheless, in future one should investigate the contributions of the minimal length when the acceleration approaches $1/l$. An eight-dimensional formulation ($X^\mu = x^\mu + I_c^L u^\mu$) should be systematically worked out!

Finally, in the last section, we discussed observable effects leading to differences between standard (Einstein's) and pseudo-complex General Relativity. We concentrated on circular orbits around a gray star. The pc-Schwarzschild case, of a non-rotating gray star, and the pc-Kerr case, of a rotating gray star, were investigated. In case of the pc-Schwarzschild solution the last stable orbit changed from $6m$ to about $5.25m$. In the pc-Kerr solution the last stable orbit is further out and the orbital frequency is always lower than in the standard Schwarzschild case, exhibiting also a maximum value, i.e., a minimal time for the orbit.

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Appendix

Reissner-Nordström solution with unchanged source

Assuming that $\Xi_{\mu}^{RN} = \Xi_{\mu}$, where the superscript "RN" refers to Reissner-Nordström, equation (25) leads to

$$\begin{aligned} e^{-\lambda_{RN}} \xi_0^{RN} &= e^{-\lambda} \xi_0 \\ e^{-\lambda_{RN}} \xi_1^{RN} &= e^{-\lambda} \xi_1 \\ \xi_2^{RN} &= \xi_2 \quad , \end{aligned} \tag{131}$$

so that

$$\xi_0^{RN} - \xi_1^{RN} = e^{\lambda_{RN} - \lambda_-} (\xi_0 - \xi_1) \tag{132}$$

follows.

Combining this with (69) results in

$$e^{-\lambda_{RN-}} = e^{-\lambda_-} - \frac{A}{R_-^2} \tag{133}$$

In the next step, we have to use (70), but since the equation was not explicitly proven within the main chapter we will do that first. Therefore we take (66) and replace ν'_{RN-} and ν''_{RN-} by using the equations (64) and (65), which gives

$$\begin{aligned} & -\lambda''_{RN-} + \frac{1}{2} (\xi_0^{RN} - \xi_1^{RN}) + \frac{R_-}{2} (\xi_0^{RN'} - \xi_1^{RN'}) \\ & \quad - \frac{\lambda'_{RN-}}{2} \left(-\lambda'_{RN-} + \frac{R_-}{2} (\xi_0^{RN} - \xi_1^{RN}) \right) \\ & \quad + \frac{1}{2} \left(-\lambda'_{RN-} + \frac{R_-}{2} (\xi_0^{RN} - \xi_1^{RN}) \right)^2 - \frac{2\lambda'_{RN-}}{R_-} \\ & \quad = \\ & \quad \xi_1^{RN} - \frac{2A}{R_-^2} e^{\lambda_{RN-}} \quad . \end{aligned} \tag{134}$$

In the next step we eliminate the big brackets

$$\begin{aligned} & -\lambda''_{RN-} + \frac{1}{2} (\xi_0^{RN} - \xi_1^{RN}) + \frac{R_-}{2} (\xi_0^{RN'} - \xi_1^{RN'}) + (\lambda'_{RN-})^2 \\ & \quad - \frac{3}{4} R_- \lambda'_{RN-} (\xi_0^{RN} - \xi_1^{RN}) + \frac{1}{8} R_-^2 (\xi_0^{RN} - \xi_1^{RN})^2 - \frac{2\lambda'_{RN-}}{R_-} \\ & \quad = \\ & \quad \xi_1^{RN} - \frac{2A}{R_-^2} e^{\lambda_{RN-}} \end{aligned} \tag{135}$$

and rearrange the terms

$$\begin{aligned}
& -\lambda''_{RN-} + \frac{1}{2}(\xi_0^{RN} - \xi_1^{RN}) + \frac{R_-}{4}(\xi_0^{RN'} - \xi_1^{RN'}) + (\lambda'_{RN-})^2 \\
& -\frac{1}{4}R_- \lambda'_{RN-} (\xi_0^{RN} - \xi_1^{RN}) - \frac{2\lambda'_{RN-}}{R_-} + \frac{2A}{R_-^4} e^{\lambda_{RN-}} - \xi_1^{RN} \\
& = \\
& -\frac{1}{4}R_- (\xi_0^{RN'} - \xi_1^{RN'}) + \frac{1}{2}\lambda_{RN-}^{RN'} R_- (\xi_0^{RN} - \xi_1^{RN}) - \frac{1}{8}R_-^2 (\xi_0^{RN} - \xi_1^{RN})^2
\end{aligned} \tag{136}$$

When we differentiate (68) and reorganize the terms in a way, so that ξ'_2 stands alone on the left side, we get

$$\begin{aligned}
\xi'_2 &= (R_- e^{-\lambda_{RN-}})'' + \frac{1}{2}R_- e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) \\
& - \frac{1}{4}R_-^2 \lambda'_{RN-} e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) + \frac{1}{4}R_-^2 e^{-\lambda_{RN-}} (\xi_0^{RN'} - \xi_1^{RN'}) + \frac{2A}{R_-^3} \\
& = -2\lambda'_{RN-} e^{-\lambda_{RN-}} - R_- \lambda''_{RN-} e^{-\lambda_{RN-}} + R_- (\lambda'_{RN-})^2 e^{-\lambda_{RN-}} \\
& + \frac{1}{2}R_- e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) \frac{1}{4}R_-^2 \lambda'_{RN-} e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) \\
& - + \frac{1}{4}R_-^2 e^{-\lambda_{RN-}} (\xi_0^{RN'} - \xi_1^{RN'}) + \frac{2A}{R_-^3} .
\end{aligned} \tag{137}$$

Multiplying with $\frac{e^{\lambda_{RN-}}}{R_-}$ and rearranging terms yields

$$\begin{aligned}
\frac{e^{\lambda_{RN-}}}{R_-} \xi'_2 &= -\lambda''_{RN-} + \frac{1}{2}(\xi_0^{RN} - \xi_1^{RN}) + \frac{R_-}{4}(\xi_0^{RN'} - \xi_1^{RN'}) + (\lambda'_{RN-})^2 \\
& -\frac{1}{4}R_- \lambda'_{RN-} (\xi_0^{RN} - \xi_1^{RN}) - \frac{2\lambda'_{RN-}}{R_-} + \frac{2A}{R_-^4} e^{\lambda_{RN-}}
\end{aligned} \tag{138}$$

Hence we get equation (70) and we can use it for the following calculations.

At first we multiply the equation with $e^{-\lambda_{RN-}}$ and use (131) to replace the ξ_μ^{RN} . So the terms on the left side can be rewritten to

$$\frac{\xi_2^{RN'}}{R_-} = \frac{\xi'_2}{R_-} \tag{139}$$

$$e^{-\lambda_{RN-}} \xi_1^{RN} = e^{-\lambda_-} \xi_1, \tag{140}$$

whereas those on the right side transform to

$$\lambda'_{RN-} e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN}) = \lambda'_{RN-} e^{-\lambda_-} (\xi_0 - \xi_1) \tag{141}$$

$$\begin{aligned}
e^{-\lambda_{RN-}} (\xi_0^{RN'} - \xi_1^{RN'}) &= e^{-\lambda_{RN-}} (e^{\lambda_{RN-} - \lambda_-} (\xi_0 - \xi_1))' \\
&= (\lambda'_{RN-} - \lambda'_-) e^{-\lambda_-} (\xi_0 - \xi_1) \\
&+ e^{-\lambda_-} (\xi'_0 - \xi'_1)
\end{aligned} \tag{142}$$

$$e^{-\lambda_{RN-}} (\xi_0^{RN} - \xi_1^{RN})^2 = e^{\lambda_{RN-} - 2\lambda_-} (\xi_0 - \xi_1)^2 . \tag{143}$$

So we get

$$\begin{aligned} \frac{\xi_2'}{R_-} - e^{-\lambda_-} \xi_1 &= \frac{R_-}{4} (\lambda'_{RN-} + \lambda'_-) e^{-\lambda_-} (\xi_0 - \xi_1) \\ &\quad - \frac{R_-}{4} e^{-\lambda_-} (\xi_0' - \xi_1') - \frac{R_-^2}{8} e^{\lambda_{RN-} - 2\lambda_-} (\xi_0 - \xi_1)^2 . \end{aligned} \quad (144)$$

Replacing ξ_1 with (28) and subtracting (38) times $e^{-\lambda_-}$ leads to

$$0 = \frac{R_-}{4} (\lambda'_{RN-} - \lambda'_-) e^{-\lambda_-} (\xi_0 - \xi_1) - \frac{R_-^2}{8} (1 - e^{\lambda_{RN-} - \lambda_-}) e^{-\lambda_-} (\xi_0 - \xi_1)^2 \quad (145)$$

Therefore we can conclude either

$$\xi_0 - \xi_1 = 0 , \quad (146)$$

which combined with the ideal fluid ansatz and $p = w\rho$ leads to a case equivalent to introducing a cosmological constant, since (27) demands $w = -1$. Or the remaining part has to vanish which yields

$$\xi_0 - \xi_1 = \frac{2 (\lambda'_{RN-} - \lambda'_-)}{R_- (e^{\lambda_{RN-} - \lambda_-} - 1)} . \quad (147)$$

In the next step we want to eliminate all Reissner-Nordström variables. Therefore we begin with replacing λ'_{RN-} and for convenience λ'_- as well

$$\begin{aligned} \lambda'_{RN-} &= - (e^{-\lambda_{RN-}})' e^{\lambda_{RN-}} = - (e^{-\lambda_-})' e^{\lambda_{RN-}} - \frac{2A}{R_-^3} e^{\lambda_{RN-}} \\ \lambda'_- &= - (e^{-\lambda_-})' e^{\lambda_-} , \end{aligned} \quad (148)$$

so that we can rewrite (147) multiplied by $\frac{e^{-\lambda_-}}{2}$ to

$$\begin{aligned} \frac{e^{-\lambda_-}}{2} (\xi_0 - \xi_1) &= \frac{- (e^{-\lambda_-})' (e^{\lambda_{RN-}} - e^{\lambda_-}) - \frac{2A}{R_-^3} e^{\lambda_{RN-}}}{R_- (e^{\lambda_{RN-}} - e^{\lambda_-})} \\ \frac{e^{-\lambda_-}}{2} (\xi_0 - \xi_1) &= - \frac{(e^{-\lambda_-})'}{R_-} - \frac{2A}{R_-^4} \frac{e^{\lambda_{RN-}}}{(e^{\lambda_{RN-}} - e^{\lambda_-})} . \end{aligned} \quad (149)$$

Now we rewrite the denominator of the second term

$$\begin{aligned} e^{\lambda_{RN-}} - e^{\lambda_-} &= \frac{1}{e^{-\lambda_-} - \frac{A}{R_-^2}} - e^{\lambda_-} \\ e^{\lambda_{RN-}} - e^{\lambda_-} &= \frac{1}{e^{-\lambda_-} - \frac{A}{R_-^2}} - \frac{1 - \frac{A}{R_-^2} e^{\lambda_-}}{e^{-\lambda_-} - \frac{A}{R_-^2}} = \frac{A}{R_-^2} e^{\lambda_-} e^{\lambda_{RN-}} , \end{aligned} \quad (150)$$

so that the equation is transformed into

$$\frac{e^{-\lambda_-}}{2}(\xi_0 - \xi_1) = -\frac{(e^{-\lambda_-})'}{R_-} - \frac{2}{R_-^2}e^{-\lambda_-} \quad . \quad (151)$$

At this point we use the ideal fluid ansatz, we replace the ξ according to (27) and $e^{-\lambda_-}$ utilizing (40)

$$\begin{aligned} \frac{8\pi\kappa}{c^2} \left(\rho + \frac{p}{c^2} \right) &= -\frac{2M_-}{R_-^3} - \frac{8\pi\kappa}{c^2 R_-^3} \int R_-^2 \rho dR_- + \frac{8\pi\kappa}{c^2} \rho \\ &\quad - \frac{2}{R_-^2} + \frac{4M_-}{R_-^3} + \frac{16\pi\kappa}{c^2 R_-^3} \int R_-^2 \rho dR_- \\ \Rightarrow \frac{8\pi\kappa}{c^2} \frac{p}{c^2} &= -\frac{2}{R_-^2} + \frac{2M_-}{R_-^3} + \frac{8\pi\kappa}{c^2 R_-^3} \int R_-^2 \rho dR_- \quad . \end{aligned} \quad (152)$$

Multiplying the equation with $\frac{c^2 R_-^3}{8\pi\kappa}$ and differentiating it leads to

$$3R_-^2 \frac{p}{c^2} + R_-^3 \frac{p'}{c^2} = -\frac{c^2}{4\pi\kappa} + R_-^2 \rho \quad (153)$$

Assuming the ansatz $\frac{p}{c^2} = w\rho$ we get

$$w\rho' = \frac{1-3w}{R_-} \rho - \frac{c^2}{4\pi\kappa R_-^3} \quad (154)$$

Now we have to distinguish between three cases. First of all let w be 0, then we can easily calculate ρ

$$\rho = \frac{c^2}{4\pi\kappa} R_-^{-2} \quad , \quad (155)$$

but with (42) we observe, that M_{de} is proportional to R_- , which leads to a constant correction within the metric (see (43)). Thus this case is unphysical.

For the other two cases we have to solve the differential equation. This is easily done by calculating the homogeneous solution ρ_h

$$\rho_h = C R_-^{\frac{1-3w}{w}} \quad (156)$$

followed by a variation of the constant

$$\rho = C(R_-) R_-^{\frac{1-3w}{w}} \quad (157)$$

$$\Rightarrow C' = -\frac{c^2}{4\pi\kappa w} R_-^{-\frac{1}{w}} \quad (158)$$

Let w be 1, then ρ is given by

$$\rho = \tilde{C} R_-^{-2} - \frac{c^2}{4\pi\kappa} \ln \left(\frac{R_-}{R_{0-}} \right) R_-^2 \quad . \quad (159)$$

Again inserting this in (42) excludes the case.

Now let us assume w is any possible value except for 0, 1 and -1 (for $w = -1$ (147)

has not to be fulfilled, since $\xi_0 = \xi_1$). In this case the variation of the constant leads to

$$\rho = \tilde{C} R_-^{\frac{1-3w}{w}} - \frac{c^2}{4\pi\kappa(w-1)} R_-^{-2} . \quad (160)$$

Hence again M_{de} has a term proportional to R_- , which however can be small, if the absolute value of w is much bigger than 1. In this case we get in (43) a correction almost of the order $\frac{1}{R_-}$, which is too high to be physical. So we showed, that the ideal fluid ansatz is inconsistent with the assumption $\Xi_\mu^{RN} = \Xi_\mu$; at least with a linear correlation between $\frac{\rho}{c^2}$ and ρ .

Furthermore even if we would allow such corrections to the metric, the inconsistency of the ideal fluid ansatz with the assumption $\Xi_\mu^{RN} = \Xi_\mu$ can be shown. However to do that cumbersome calculations are needed, in which the solution for ρ is inserted into the equation for p' in (47) and the orders of R_- are compared.

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